General Relativity Exam Problem 4

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AMATH 875

Problem Statement

Give an example of two isometric (pseudo) Riemannian manifolds.

Recall

- Let (M, g) and (N, h) be two (pseudo) Riemannian manifolds.
- Let $f : M \to N$ be a diffeomorphism.

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Definition (Isometry)

If f satisfies $f^*h = g$, then f is called an isometry.

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Setup

Take $M = N = \mathbb{R}^n$, and equip M with the standard Euclidean metric $g = \langle - | - \rangle$, and N with $h = \langle A \cdot - | A \cdot - \rangle$ where $A \in \mathsf{SO}(n)$ is a rotation.

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- a and b are arbitrary, thus $f^*h = g$, and f is an isometry (as expected).

Extensions

- This works for any $A \in \mathsf{GL}(n\,;\mathbb{R})$ where we define f by $f(v) = A^{-1} \cdot v$ instead of transpose.
- Can also be generalized to flows on manifolds, and often Lie groups[1] generate flows.

[1] Manifolds with a group structure that is compatible with the differential structure.