General Relativity for Cosmology Lectures 2

Name: Nate Stemen (20906566)
Email: nate@stemen.email
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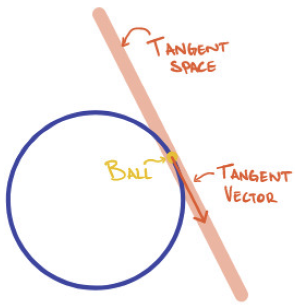
For me, there were three main takeaways from this lecture:

1. The ideas underlying the tangent space at a point
2. The (algebraic) definition and construction of the tangent space at a point
3. The way in which a diffeomorphism of smooth manifolds induces a vector space isomorphism of tangent spaces

Item 1: If we think of a ball travelling on a circle we use the fact that the circle's tangent vectors coincide with the ball's velocity vectors. Indeed depending on the



velocity of the the tangent plane to the circle will always contain it's velocity vector and it will also be a vector space. This idea can of course be generalized to more complicated spaces than circles and spheres and indeed that's exactly what we set out to do.

Item 2: This was the most fascinating part of the lecture to me as I've had a picture of the tangent space in my head, but never seen the formal construction using derivations. What we do is look at all functions that are equal "around" our point of interest p and make an equivalence class—in effect only leaving functions who behave differently around p. We then define the tangent space $T_p(M)$ of our manifold M at p to be the vector space of derivations[1] on our equivalence class of functions.

Item 3: The last key takeaway involves two manifolds M and N and a diffeomorphism (differentiable structure preserving) map $\phi : M \to N$. If we wanted to learn about $T_q(N)$ given knowledge about $T_p(M)$ and $q = \phi(p)$, then we can do so via an induced map $T_p\phi$ which is built up from the induced algebra homomorphism of the germ-spaces[2] of each manifold. Honestly this part I'm a little iffy about, but I'm trying.

[1] This space remains to be proven this is truly a vector space, but it does indeed work out.
[2] I'm not really sure what to call these $\mathcal{F}(p)$ spaces.