Lie Groups and Lie Algebras Final Assessment

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1

Recall that the center of a matrix Lie group G is by definition $Z(G) := \{g \in G : gh = hg \text{ for all } h \in G\}$. Below we let $n \geq 2$.

(a) With the help of Schur's lemma, determine the centers of $\mathsf{U}(n)$ and $\mathsf{SU}(n)$. Deduce that $\mathsf{U}(n)$ and $\mathsf{SU}(n) \times \mathsf{U}(1)$ are not isomorphic as Lie groups.

(b) Prove that $\mathsf{U}(n)$ and $\mathsf{SU}(n) \times \mathsf{U}(1)$ nonetheless have isomorphic Lie algebras.

Solution. (a) Let $A \in Z(\mathsf{U}(n))$, then by the definition of $\mathsf{U}(n)$ we can view A as a map taking $\mathsf{U}(n)$ to itself $\mathsf{U}(n) \xrightarrow{A} \mathsf{U}(n)$. Let $\rho : \mathsf{U}(n) \to \mathsf{GL}(n; \mathbb{C})$ be the representation defined by $\rho(X) = X$. Then A intertwines ρ:

$$A \circ \rho(X) = A(\rho(X)) = AX = XA = \rho(X) \circ A.$$

By Schur's lemma A must then either be 0 or a scalar multiple of the identity $\mathbb{1}$. The unitary group does not include 0 so $A = \lambda \mathbb{1}$, and in addition

$$AA^\dagger = \lambda\overline{\lambda}\mathbb{1} = \mathbb{1} \implies \lambda = e^{i\varphi}.$$

Thus the center of $\mathsf{U}(n)$ is $\{e^{i\varphi}\mathbb{1} : \varphi \in \mathbb{R}\}$ which is isomorphic to S^1 the circle group.

The above argument applies to $\mathsf{SU}(n)$ as well, but we have the extra condition that $\det A = 1$. By properties of the determinant[1] we know $\det A = e^{in\varphi} = 1$ so the center of $\mathsf{SU}(n)$ is the group of n-th roots of unity. The n-th roots of unity are known to be isomorphic to $\mathbb{Z}/n\mathbb{Z}$, so we will say $Z(\mathsf{SU}(n)) = \mathbb{Z}/n\mathbb{Z}$.

Further we have $Z(\mathsf{SU}(n) \times \mathsf{U}(1)) = \mathbb{Z}/n\mathbb{Z} \times S^1$ which is clearly not isomorphic to S^1. Now if $\mathsf{U}(n)$ was isomorphic to $\mathsf{SU}(n) \times \mathsf{U}(1)$, then their centers would also be isomorphic, but they are not, so they cannot be isomorphic as groups, and hence Lie groups.

(b) First we define the following function $\Phi : \mathsf{SU}(n) \times \mathsf{U}(1) \to \mathsf{U}(n)$:

$$\Phi(X, \alpha) := \alpha X.$$

This is surely an element of $\mathsf{U}(n)$ because $\alpha X(\alpha X)^\dagger = \alpha\overline{\alpha}XX^\dagger = \mathbb{1}$. It is also a group homomorphism

$$\Phi(X, \alpha)\Phi(Y, \beta) = \alpha X \beta Y = \alpha\beta XY = \Phi(XY, \alpha\beta)$$

and is clearly continuous since we're just doing scalar multiplication. Thus Φ is a Lie group homomorphism. To see this map is a surjection, take $X \in \mathsf{U}(n)$ and we will find a pair $(Y, \alpha) \in \mathsf{SU}(n) \times \mathsf{U}(1)$ that hits it. Some inspection yields

$$Y = \frac{X}{(\det X)^{1/n}} \qquad\qquad \alpha = 1.$$

[1] $\det(\alpha A) = \alpha^n \det A$ if $A \in \mathcal{M}_n(\mathbb{F})$.

Thus Φ is a surjection. We now show Φ also has a discrete kernel. If $\Phi(X, \alpha) = \alpha X = \mathbb{1}$, then first X must be diagonal, and hence also in the center of $\mathsf{SU}(n)$. As we found above $Z(\mathsf{SU}(n)) \cong \mathbb{Z}/n\mathbb{Z}$, and hence $\ker \Phi$ is discrete.

By Proposition 3.31 of Hall (page 63), we have $\text{Lie}(\ker \Phi) = \ker \phi$ where ϕ is a Lie algebra homomorphism between the Lie algebras of $\mathsf{SU}(n) \times \mathsf{U}(1)$ and $\mathsf{U}(n)$. However, the Lie algebra of a discrete group is trivial, and hence $\ker \phi$ is trivial which implies that ϕ is an isomorphism of Lie algebras.

2

(a) Prove that every element of SO(3) except the identity belongs to exactly one maximal torus in SO(3).

(b) Consider the following maximal torus in SO(3):

$$T = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Prove that it's Weyl group is isomorphic to the finite abelian group $\mathbb{Z}_2$.

Solution. (a) Using the maximal torus $\leftrightarrow$ Cartan subalgebra correspondence we can pass to so(3) and rephrase the problem as "prove that every vector of so(3) except the zero vector belongs to exactly one maximal torus in so(3)." Here we have the following basis

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

with the following commutation relations:

$$[e_1, e_2] = -e_3 \qquad [e_1, e_3] = e_2 \qquad [e_2, e_3] = -e_1.$$

Since so(3) is not commutative there are no 3-dimensional Cartan subalgebras. The commutation relations also show there are no 2-dimensional Cartan subalgebras since we never have $[e_i, e_j] = e_i$. Hence all the Cartan subalgebras are 1-dimensional, and because they must span so(3), they only intersect at the origin. We can now exponentiate these Cartan subalgebras to obtain maximal tori which also only intersect at $\exp\left(\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \mathbb{1}_3$.

The first way I thought about this problem was to show that rotations around the 3-axis are all maximal tori, but I couldn't figure out how to show they were the *only* maximal tori.

(b) First note that $T \subseteq N(T)$ almost by definition. Now to calculate what else is in $N(T)$ we need a general paremetrization of elements in SO(3) and for that we look to the Euler angle decomposition. In particular if $R_z(\theta)$ is a rotation of angle θ about the z-axis, then any element $A \in \mathrm{SO}(3)$ can be written as

$$A = R_z(\alpha) R_x(\beta) R_z(\gamma).$$

Thus we need to figure out for which $\alpha, \beta, \gamma \in [0, 2\pi)$ does $A R_z(\theta) A^\intercal = R_z(\phi)$. Doing the algebra by hand is possible, but thankfully we have computers.

```
import sympy as sym
from sympy.abc import alpha, beta, gamma, theta
from sympy.matrices import rot_axis1, rot_axis3

r = rot_axis3(alpha) * rot_axis1(beta) * rot_axis3(gamma)
normalized = sym.simplify(r * rot_axis3(theta) * r.T)
```

Now we have `normalized`[2] computed as $AR_z(\theta)A^\intercal$ and we know it should look like another z-rotation. Thus $[A]_{33} = 1$ because z-rotations fix the z-axis. We can then access the $[A]_{33}$ element with `normalized[8]`:

```
print(normalized[8])
```

$$\sin^2(\beta)\cos(\theta) - \sin^2(\beta) + 1 \tag{1}$$

Since this must be equal to 1 for all θ we must have $\beta = 0, \pi$. If $\beta = 0$, then A is purely a rotation around the z-axis which we already knew was in $N(T)$. In the case $\beta = \pi$, we can take $\alpha = 0 = \gamma$ to conclude the following matrix is in $N(T)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2}$$

Since this is clearly not in T (it doesn't have a $+1$ in the bottom right hand corner). Indeed these β are the only possible values because of eq. (1), and we can verify they work with the computer.

```
print(normalized.subs( { beta: sym.pi } ))
```

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Indeed we have the same thing with $\beta = 0$, but that can be seen more easily from $A = R_z(\alpha)R_x(\beta = 0)R_z(\gamma) = R_z(\alpha + \gamma)$.

Thus the Weyl group $N(T)/T$ contains two elements: one being the identity $[\mathbb{1}]$, and the other from eq. (2) which we will call $[x]$. They have the following multiplicative structure:

$$[\mathbb{1}] \cdot [\mathbb{1}] = [\mathbb{1}] \qquad [\mathbb{1}] \cdot [x] = [x] \qquad [x] \cdot [x] = \mathbb{1}$$

This is exactly the structure of $\mathbb{Z}/2\mathbb{Z}$ under the map $[\mathbb{1}] \to [0]$ and $[x] \to [1]$.

[2]I'll include a page with the entire matrix printed, but it's too big (and not very helpful) to put here.

3

Recall the SU(2)-representation $V_m(\mathbb{C}^2)$. For $0 \leq k \leq m$ we write $f_k(z_1, z_2) = z_1^k z_2^{m-k}$. Prove the following defines an SU(2) invariant inner product on $V_m(\mathbb{C}^2)$:

$$\left(\sum_{k=0}^{m} \alpha_k f_k, \sum_{l=0}^{m} \beta_l f_l \right) = \sum_{k=0}^{m} k!(m-k)! \alpha_k \overline{\beta_k}.$$

Solution. First such an invariant product must exist by Weyl's unitarian[3] trick. Suppose $\langle - | - \rangle$ is an inner product on $V_m(\mathbb{C}^2)$, then we can define a new inner product by averaging over all the elements of SU(2):

$$(f, h) := \int_{\mathsf{SU}(2)} \langle \Pi_m(g) \cdot f \mid \Pi_m(g) \cdot h \rangle \mathrm{d}\mu(g) \tag{3}$$

This new inner product is manifestly SU(2) invariant, and as we proved in Assignment 3 problem 6(b), the invariant inner product is unique up to scaling by a positive constant.

I've written eq. (3) assuming we have a normalized Haar measure that can be defined for all compact[4] Lie groups. In particular it is defined to be left invariant, but it is also right invariant.

Perhaps another way to see such an inner product exists is to use the isomorphism we proved in assignment 1 problem 6(c). That is $\mathsf{SU}(2) \cong \mathsf{Sp}(1)$. Indeed the action of unit quaternions is just a rotation, and simultaneously rotating vectors inside an inner product does not affect the value.

Now to see if this inner product defined above is *actually* SU(2)-invariant we need to test if $(\Pi_m(g) \cdot f, \Pi_m(g) \cdot h) = (f, h)$ for all $g \in \mathsf{SU}(2)$. Our first step will be to writen out $\Pi_m(g) \cdot f$ for an arbitrary element $f \in V_m(\mathbb{C}^2)$. Here we use the following parametrization of elements $g \in \mathsf{SU}(2)$: $g = \left[\begin{smallmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{smallmatrix}\right]$.

$$\begin{aligned}
\Big[\Pi_m(g) \cdot f\Big](z) &= f\left(g^{-1} \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]\right) \\
&= f\left(\begin{bmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) \\
&= \sum_{k=0}^{m} a_k \left(\overline{\alpha} z_1 + \overline{\beta} z_2\right)^k \left(-\beta z_1 + \alpha z_2\right)^{m-k} \\
&= \sum_{k,l,p=0}^{m,k,m-k} a_k \binom{k}{l} \binom{m-k}{p} (-1)^{m-k-p} \alpha^p \overline{\alpha}^{k-l} \beta^{m-k-p} \overline{\beta}^l z_1^{m-l-p} z_2^{l+p}
\end{aligned}$$

From here I'd really like to be able to write this as $\sum_{n=0}^{m} \tilde{a}_n f_n$, but I cannot find a way. I can see that the powers of z_1 and z_2 are intimately tied to $l + p$, but I cannot figure out how to get rid of the l's and p's.

If I was able to do this, I would then take $\sum_{k=0}^{m} k!(m-k)! \tilde{a}_k \overline{\tilde{b}_k}$ and show it's equal to $\sum_{k=0}^{m} k!(m-k)! a_k \overline{b_k}$. Easier said than done though.

I also tried acting $A \in \mathsf{SU}(2)$ on a single basis element f_k to try and make this easier, but I *still* couldn't get anywhere.

[3]Is it unitarian or unitary?

[4]And maybe locally compact as well?

4

Define $V_{1,1}(\mathbb{C}^3) := \text{span}_{\mathbb{C}} \left\{ z_i\overline{z_j} : i,j \in \{1,2,3\} \right\}$. For $f \in V_{1,1}(\mathbb{C}^3)$ and $A \in \text{SU}(3)$ we let $(\Pi_{1,1}(A) \cdot f)(z) := f(A^{-1}z)$. Also, let $\triangle \equiv \sum_{j=1}^{3} \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}}$ and define $\mathcal{H}_{1,1}(\mathbb{C}^3) := \left\{ f \in V_{1,1}(\mathbb{C}^3) : \triangle f = 0 \right\}$.

(a) Prove that $(\Pi_{1,1}, V_{1,1}(\mathbb{C}^3))$ is a complex representation of SU(3) and that $\mathcal{H}_{1,1}(\mathbb{C}^3)$ is an invariant subspace.

(b) Consider the associated $\mathfrak{sl}(3;\mathbb{C})$-representation on $\mathcal{H}_{1,1}(\mathbb{C}^3)$. Prove that this is irreducible, and find it's dimension and highest weight.

Solution. (a) First let's the action of $\Pi_{1,1}(A)$ does indeed bring us back into $V_{1,1}(\mathbb{C}^3)$. We'll represent an arbitrary vector $f \in V_{1,1}(\mathbb{C}^3)$ as $f = \sum_{i,j=1}^{3} a_{ij}z_i\overline{z_j}$.

$$\left[\Pi_{1,1}(A) \cdot f\right](z) = f(A^{-1}z)$$
$$= \sum_{i,j=1}^{3} a_{ij}\left(a_{i1}^{-1}z_1 + a_{i2}^{-1}z_2 + a_{i3}^{-1}z_1\right)\left(\overline{a_{j1}^{-1}}\overline{z_j} + \overline{a_{j2}^{-1}}\overline{z_2} + \overline{a_{j3}^{-1}}\overline{z_3}\right)$$

Written out this way it's not hard to see this is again an element of $V_{1,1}(\mathbb{C}^3)$. To see this map is also a Lie group homomorphism we can look at the action of A and B on f.

$$\left(\Pi_{1,1}(A) \cdot \left[\Pi_{1,1}(B) \cdot f\right]\right)(z) = \left[\Pi_{1,1}(B) \cdot f\right](A^{-1}z)$$
$$= f(B^{-1}A^{-1}z)$$
$$= \left[\Pi_{1,1}(AB) \cdot f\right](z)$$

That $\Pi_{1,1}$ is continuous follows from the fact that matrix inversion is continuous.

Now we compute what $\mathcal{H}_{1,1}(\mathbb{C}^3)$ looks like with respect to $\triangle$ defined above. I'll use $\partial_i \equiv \frac{\partial}{\partial z_i}$ and $\partial_{\overline{i}} \equiv \frac{\partial}{\partial \overline{z_i}}$. We have

$$\partial_k\partial_{\overline{k}} \sum_{i,j=1}^{3} a_{ij}z_i\overline{z_j} = \partial_k \sum_{i,j=1}^{3} a_{ij}z_i\delta_{jk} = \partial_k \sum_{i=1}^{3} a_{ik}z_i = a_{kk}$$

and thus for a general element f we have

$$\triangle f = \sum_{i=1}^{3} a_{ii}$$

and thus

$$\mathcal{H}_{1,1}(\mathbb{C}^3) = \left\{ f \in V_{1,1}(\mathbb{C}^3) : \sum a_{ii} = 0 \right\}.$$

To show this is an invariant subspace we will show that the action of $U \in \text{SU}(3)$

commutes with the action of the Laplacian $\triangle$.

$$\begin{aligned}
\partial_{\bar{i}}(f(Uz)) &= \sum_{k=1}^{3} \partial_{\bar{i}} f(Uz)\overline{u_{ki}} \\
\partial_i\partial_{\bar{i}}(f(Uz)) &= \sum_{k,l=1}^{3} \partial_i\partial_{\bar{i}} f(Uz)\overline{u_{ki}}u_{li} \\
\triangle[f(Uz)] &= \sum_{i,k,l=1}^{3} \partial_i\partial_{\bar{i}} f(Uz)\delta_{kl} = (\triangle f)(Uz)
\end{aligned}$$

Thus if $\triangle f(z) \equiv 0$, then $\triangle f(Uz) = 0$.

(b) The above representation of $\mathsf{SU}(3)$ gives us a representation of the Lie algebra $\mathsf{su}(3)$, and by complex linearity a representation on the complexification $\mathsf{su}(3)_{\mathbb{C}} \cong \mathsf{sl}(3;\mathbb{C})$ which we can calculate with

$$\pi_{1,1}(X) := \frac{\mathrm{d}}{\mathrm{d}t}\,\Pi_{1,1}(\mathrm{e}^{tX})\bigg|_{t=0}.$$

Acting on an element $f \in V_{1,1}(\mathbb{C}^3)$ and $z \equiv z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}$ we have

$$\begin{aligned}
\Big[\pi_{1,1}(X)\cdot f\Big](z) &= \frac{\mathrm{d}}{\mathrm{d}t}\, f(\mathrm{e}^{-tX}z)\bigg|_{t=0} \\
&= \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial t}\bigg|_{t=0} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial t}\bigg|_{t=0} + \frac{\partial f}{\partial z_3}\frac{\partial z_3}{\partial t}\bigg|_{t=0} \\
&= -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2 + X_{13}z_3) \\
&\quad - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2 + X_{23}z_3) \\
&\quad - \frac{\partial f}{\partial z_3}(X_{31}z_1 + X_{32}z_2 + X_{33}z_3)
\end{aligned}$$

We can now calculate $\pi_{1,1}(X)$ for every basis element of $\mathsf{sl}(3;\mathbb{C})$ using the basis defined in Hall on page 142.

$$\pi_{1,1}(H_1) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} \qquad \pi_{1,1}(H_2) = -z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}$$

$$\begin{aligned}
\pi_{1,1}(X_1) &= -z_2\frac{\partial}{\partial z_1} & \pi_{1,1}(X_2) &= -z_3\frac{\partial}{\partial z_2} & \pi_{1,1}(X_3) &= -z_3\frac{\partial}{\partial z_1} \\
\pi_{1,1}(Y_1) &= -z_1\frac{\partial}{\partial z_2} & \pi_{1,1}(Y_2) &= -z_2\frac{\partial}{\partial z_3} & \pi_{1,1}(Y_3) &= -z_1\frac{\partial}{\partial z_3}
\end{aligned}$$

Now, H_1 and H_2 applied to an arbitrary element of $\mathcal{H}_{1,1}(\mathbb{C}^3)$:

$$\begin{aligned}
\pi_{1,1}(H_1)\cdot f &= -a_{11}z_1\overline{z_1} - a_{12}z_1\overline{z_2} - a_{13}z_1\overline{z_3} + a_{21}z_2\overline{z_1} + a_{22}z_2\overline{z_2} + a_{23}z_2\overline{z_3} \\
\pi_{1,1}(H_2)\cdot f &= -a_{21}z_2\overline{z_1} - a_{22}z_2\overline{z_2} - a_{23}z_2\overline{z_3} + a_{31}z_3\overline{z_1} + a_{32}z_3\overline{z_2} + a_{33}z_3\overline{z_3}
\end{aligned}$$

Vector v	Weight μ
$a_{11}z_1\overline{z_1} + a_{12}z_1\overline{z_2} + a_{13}z_1\overline{z_3}$	$(-1,0)$
$a_{21}z_2\overline{z_1} + a_{22}z_2\overline{z_2} + a_{23}z_2\overline{z_3}$	$(1,-1)$
$a_{31}z_3\overline{z_1} + a_{32}z_3\overline{z_2} + a_{33}z_3\overline{z_3}$	$(0,1)$

Table 1: Weight Spaces for $\mathfrak{h} = \text{span}(H_1, H_2)$

Where we have to remember a_{33} can be written as $-(a_{11} + a_{22})$ because f is harmonic. From the action of H_1 and H_2 we can see we have three weight spaces. Since we are looking for an *irreducible* representation we will look at the vector subspace with a weight composed of *non-negative* integers. We now check if the associated vector gives us the satisfying conditions to make this representation one of highest weight cyclic. It's not hard to see because $\pi_{1,1}(X_i)$ only involves the partial derivatives of z_1 and z_2, that all of $\pi_{1,1}(X_i) \cdot v = 0$ for all $i = 1, 2, 3$.

We now apply $\pi_{1,1}(Y_i)$ to this vector to see if v "generates" the entire vector space.

$$\begin{aligned}
\pi_{1,1}(Y_1) \cdot v &= 0 \\
\pi_{1,1}(Y_2) \cdot v &= -a_{31}z_2\overline{z_1} - a_{32}z_2\overline{z_2} - a_{33}z_2\overline{z_3} \\
\pi_{1,1}(Y_3) \cdot v &= -a_{31}z_1\overline{z_1} - a_{32}z_1\overline{z_2} - a_{33}z_1\overline{z_3}
\end{aligned}$$

Thus the only invariant subspace containing v is $\mathcal{H}_{1,1}(\mathbb{C}^3)$ itself, and hence this representation is highest weight cyclic. We now use two facts from Hall, and one from class:

Fact 1 (Theorem 10.9, Hall page 275)**.** *Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.*

Fact 2 (Lecture 65)**.** $\mathfrak{sl}(n;\mathbb{C})$ *is semisimple.*

Fact 3 (Proposition 6.14, Hall page 150)**.** *Suppose* (π, V) *is a* completely reducible *representation of* $\mathfrak{sl}(3;\mathbb{C})$ *that is also a highest weight cyclic. Then* π *is irreducible.*

Combining all these we see $(\pi_{1,1}, \mathcal{H}_{1,1}(\mathbb{C}^3))$ is indeed irreducible with dimension 8 and highest weight $(0,1)$.

I'll be honest this *feels* wrong (like it should have a higher highest weight), but I really can't find anything wrong.

5

Recall the following basis for $\mathfrak{sl}(2;\mathbb{R})$ (also for $\mathfrak{sl}(2;\mathbb{C})$):

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(a) Prove that $\mathrm{ad}_X : \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{sl}(2;\mathbb{C})$ is not diagonalizable, while $\mathrm{ad}_A : \mathfrak{gl}(n;\mathbb{C}) \to \mathfrak{gl}(n;\mathbb{C})$ is diagonalizable for all $A \in \mathfrak{u}(n)$. Deduce that $\mathfrak{u}(n)$ contains no subalgebra isomorphic to $\mathfrak{sl}(2;\mathbb{R})$.

(b) Prove that the only ideals of $\mathfrak{sl}(2;\mathbb{R})$ are $\{0\}$ and $\mathfrak{sl}(2;\mathbb{R})$. Deduce from this and part (a) that there are no non-trivial Lie algebra homomorphisms from $\mathfrak{sl}(2;\mathbb{R})$ into $\mathfrak{u}(n)$.

(c) Deduce from parts (a) and (b) that every finite-dimensional unitary complex representation (Π, V) of $\mathsf{SL}(2;\mathbb{R})$ is trivial in the sense that $\Pi(A) = \mathrm{id}_V$ for all $A \in \mathsf{SL}(2;\mathbb{R})$. You may take for granted that $\mathsf{SL}(2;\mathbb{R})$ is connected.

Solution. (a) Since the action of ad_X is determined entirely on the action on the basis elements we recall those here:

$$\mathrm{ad}_X(H) = -2X \qquad \mathrm{ad}_X(X) = 0 \qquad \mathrm{ad}_X(Y) = H \tag{4}$$

Since ad_X is a linear operator on $\mathfrak{sl}(2;\mathbb{C})$ we can write it as a 3×3 complex matrix if we use the following isomorphism:

$$Y \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad H \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad X \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

With this we can treat $\mathfrak{sl}(2;\mathbb{C})$ like $\mathbb{C}^3$ and hence can compute the matrix representation of ad_X. Indeed a short calculation yields

$$\mathrm{ad}_X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}. \tag{5}$$

To see if this matrix is diagonalizable (which maybe you can just tell, but spelling it out more carefully was helpful for me) we can use the following fact.

Fact 4. *A matrix is diagonalizable if and only if for every eigenvalue, it's corresponding algebraic and geometric multiplicity are equal.*

Calculating the characteristic polynomial of eq. (5) we find

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} - \lambda \mathbb{1}_3 \right) = -\lambda^3$$

Hence the only eigenvalue is 0, but eq. (4) shows the only eigenvector with eigenvalue 0 is X. Clearly the eigenspace spanned by X is not three dimensional despite having algebraic multiplicity 3. Thus ad_X is diagonalizable.

To see that ad_A is diagonalizable we will use another fact.

Fact 5. *Every skew-hermitian matrix is diagonalizable.*

Now let $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ with $1 \leq i \leq n$. Even though $\mathbf{x}_i$ might equal $\mathbf{x}_j$ we can still have n-eigenvectors if they repeat. In particular we can then construct the matrices $\mathbf{x}_i\mathbf{x}_j^\dagger$. Applying ad_A to these we have

$$\mathrm{ad}_A(\mathbf{x}_i\mathbf{x}_j^\dagger) = A\mathbf{x}_i\mathbf{x}_j^\dagger - \mathbf{x}_i\mathbf{x}_j^\dagger A = (A\mathbf{x}_i)\mathbf{x}_j^\dagger + \mathbf{x}_i(A\mathbf{x}_j)^\dagger = (\lambda_i + \overline{\lambda_j})\mathbf{x}_i\mathbf{x}_j^\dagger.$$

Where we now have another eigenvalue $\lambda_i + \overline{\lambda_j}$. Since ad_A is a linear operator it can be though of as a matrix of size $n^2 \times n^2$, and we just found n^2 (possibly) distince eigenvectors/values. Hence ad_A is diagonalizable.

(b) Let $\mathsf{i} \subseteq \mathsf{sl}(2;\mathbb{R})$ be our nonempty ideal. Since it is nonempty it must contain some element of the form

$$\alpha H + \beta X + \gamma Y. \tag{6}$$

Since i is an ideal we must have $[\mathsf{sl}(2;\mathbb{R}), \mathsf{i}] \subseteq \mathsf{i}$, and in particular we can look at eq. (6) under ad_H.

$$[H, \alpha H + \beta X + \gamma Y] = \beta[H, X] + \gamma[H, Y] = 2\beta X - 2\gamma Y.$$

Thus there are two cases we must consider: $\alpha = 0$ and $\alpha \neq 0$ where i contains H.

If $H \in \mathsf{i}$, then by definition of an ideal we must have $[X, H] = -2X \in \mathsf{i}$ and $[Y, H] = 2Y \in \mathsf{i}$ and thus $\mathsf{i} = \mathsf{sl}(2;\mathbb{R})$.

If $\alpha = 0$ and the ideal i only contains elements of the form $\beta X + \gamma Y$, we again must have $[\mathsf{sl}(2;\mathbb{R}), \mathsf{i}] \subseteq \mathsf{i}$ by definition. In particular we can choose to take $[X, 2\beta X - 2\gamma Y] = -2\gamma H$, and hence if $\gamma \neq 0$ then i must contain H. We can then use the argument as above to conclude the ideal is equal to the entire group. If $\gamma = 0$, and the ideal only contains βX, we can look at $[Y, \beta X] = -\beta H$ and again if $\beta \neq 0$ then the ideal contains H. Thus if all $\alpha, \beta, \gamma = 0$, then $\mathsf{i} = \{0\}$. Thus we conclude $\mathsf{sl}(2;\mathbb{R})$ only contains trivial ideals.

If $\phi : \mathsf{sl}(2;\mathbb{R}) \to \mathsf{u}(n)$ is a Lie algebra homomorphism, then we know from class that $\ker\phi \subseteq \mathsf{sl}(2;\mathbb{R})$ must be an ideal. From above we know we only have two options for this ideal. First $\ker\phi = \mathsf{sl}(2;\mathbb{R})$ in which case *everything* gets sent to 0 and is trivial. In the second case $\ker\phi = \{0\}$, and by one of the isomorphism theorems we have

$$\mathsf{sl}(2;\mathbb{R})/\ker\phi \cong \mathsf{sl}(2;\mathbb{R}) \cong \mathrm{im}\,\phi \subseteq \mathsf{u}(n).$$

But as we've shown above $\mathsf{u}(n)$ does not contain any subalgebra isomorphic to $\mathsf{sl}(2;\mathbb{R})$ and hence we conclude the only Lie algebra homomorphism from $\mathsf{sl}(2;\mathbb{R})$ into $\mathsf{u}(n)$ is the 0 map.

(c) Suppose $\Pi : \mathsf{SL}(2;\mathbb{R}) \to \mathsf{U}(\dim V)$ is a unitary representation, that is $\Pi(A)^\dagger = \Pi(A)$ for all $A \in \mathsf{SL}(2;\mathbb{R})$. Using

$$\pi(X) := \frac{\mathrm{d}}{\mathrm{d}t}\,\Pi(\mathrm{e}^{tX})\bigg|_{t=0}$$

we can pass to the associated Lie algebra representation $\pi : \mathsf{sl}(2;\mathbb{R}) \to \mathsf{u}(\dim V)$, and by complex linearity of π we can pass to the associated representation of the complexification of $\mathsf{sl}(2;\mathbb{R})_{\mathbb{C}}$ as $\pi_{\mathbb{C}} : \mathsf{sl}(2;\mathbb{C}) \to \mathsf{u}(\dim V)$. However, as shown above the only Lie algebra homomorphism from $\mathsf{sl}(2;\mathbb{C})$ into $\mathsf{u}(\dim V)$ is trivial. Traversing back up the chain to Π we find that Π must also be trivial, and in order to satisfy the fact that it is a representation it must send the identity to the identity. Therefore

$\Pi(A) = \mathrm{id}_V$. Note here we used the fact that $\mathsf{SL}(2; \mathbb{R})$ is connected to go to and from the Lie group and Lie algebra representation. In particular because we need that Π and π to have "the same" invariant subspaces.

Also, Proposition 4.8 in Hall states "If G is connected and $\pi(X)$ is skew self-adjoint for all $X \in \mathfrak{g}$, then Π is unitary" which is exactly the case we have here. Although in our case $\pi_{\mathbb{C}}$ is only skew self-adjoint trivially.