Logic and Computability Assignment 3

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Problem 1

Let $S = \{0, 1, +, \cdot\}$.

(a) Let $\mathcal{A} = (A, \mathfrak{a})$ be an S-structure and suppose that $\sigma : A \to A$ is a bijection such that the following hold:

- $\sigma(c^A) = c^A$ for all constant symbols;
- $\sigma(f^A(a_1, \dots, a_n)) = f^A(\sigma(a_1), \dots, \sigma(a_n))$ for all n-ary function symbols f and all $a_1, \dots, a_n \in A$;
- $R^A a_1 \cdots a_n \iff R\sigma(a_1) \cdots \sigma(a_n)$ for all n-ary relation symbols R and $a_1, \dots, a_n \in A$.

Show that if $Q \subseteq A^n$ is a relation that is definable in $\mathcal{A}$ then $Qa_1 \cdots a_n \iff Q\sigma(a_1) \cdots \sigma(a_n)$. (Hint: isomorphism!)

(b) Show that the binary relation $<$ is definable in $\mathbb{R}$ (where we interpret $0, 1, +, \cdot$ as usual).

(c) Show that the binary relation $<$ is definable in $\mathbb{Z}$ (where we interpret $0, 1, +, \cdot$ as usual). (Hint: use Lagrange's four-squares theorem.)

(d) Show that the binary relation $<$ is not definable in $\mathbb{Q}(\sqrt{2})$, when we interpret $0, 1, +, \cdot$ as usual. (Hint: consider the map σ that sends $a + b\sqrt{2} \to a - b\sqrt{2}$ for $a, b \in \mathbb{Q}$.)

Solution. (a) (b) Let $i : \mathbb{R} \to \mathbb{R}$ be the function satisfying $a + i(a) = 0$: that is the additive inverse function. Now consider $\phi \in L^S$ to be given by

$$\phi = \exists r \left(v_1 + i(v_0) \equiv r^2 \wedge \neg[v \equiv 0] \right).$$

Written in the the context of $\mathbb{R}$ we have $v_1 - v_0 = r^2$ and hence $\phi[a, b] = b - a = r^2$. Hence we've found a formula that holds in $\mathbb{R}$ if and only if $a < b$. More formally we have $\mathcal{A} \vDash \phi[a, b] \iff a < b$.

(c) Let $i : \mathbb{Z} \to \mathbb{Z}$ be the function satisfying $a + i(a) = 0$: that is, it's the additive inverse function. Consider $\phi \in L^S$ to be given by

$$\phi = \exists a_0 \, \exists a_1 \, \exists a_2 \, \exists a_3 \, (v_1 + i(v_0) \equiv a_0^2 + a_1^2 + a_2^2 + a_3^2).$$

Then for ϕ to be true, we must have $v_1 + i(v_0) = v_1 - v_0$ be a positive integer. Written differently, $\mathcal{A} \vDash \phi[a, b] \iff a < b$ and hence $<$ is definable in $\mathbb{Z}$.

(d)

Problem 2

For the following we work with a first order alphabet with a binary relation R, a binary function f, an $(r+1)$-ary function g (where r is a nonnegative integer), and a constant symbol c. Write all of the formulas below in terms of formulas not involving substitutions.

(a)

$$[\forall v_0 \exists v_1 ((R v_0 v_1 \wedge R v_0 v_2) \vee \exists v_2 f v_2 v_2 \equiv v_0)] \frac{v_0}{v_0} \frac{f v_0 v_1}{v_2}$$

(b)

$$[(\exists v_0 \forall v_1 f c v_1 \equiv v_0 \to \forall v_2 f v_2 v_0 \equiv c)] \frac{f v_0 v_2}{v_2} \frac{f c v_1}{v_1} \frac{v_3}{v_0}$$

(c)

$$[(v_0 \equiv v_1 \wedge \forall v_0 R f v_2 v_1 c)] \frac{f v_2 c}{v_0}$$

(d)

$$\begin{aligned}[\forall v_0 \forall v_1 \cdots \forall v_r (v_0 \equiv v_{r+1} \wedge v_1 \equiv v_{r+2} \wedge \cdots \wedge v_r \equiv v_{2r+1} \\ \to g v_0 \cdots v_r \equiv c)] \frac{c}{v_{r+1}} \frac{c}{v_{r+2}} \cdots \frac{c}{v_{2r+1}}\end{aligned}$$

(e)

$$[\forall v_0 \exists v_0 f v_0 v_0 \equiv c] \frac{f v_0 v_0}{v_0}$$

Solution. (a) (b) (c) (d) (e)

Problem 3

Let S be a first-order alphabet with a binary function symbol f and a unary relation symbol R. Let ϕ and ψ be formulas in $\mathcal{L}^S$ in which x, y are free variables with $x \neq y$. Give formulas involving ϕ, ψ, variables, symbols from S, (,), $\forall, \exists, \wedge, \vee, \rightarrow, \leftrightarrow, \neg$, and substitution that say the following when interpreted in a domain of discourse $\mathcal{J} = (A, \mathfrak{a}, \beta)$:

(a) There are at most three values of a in A for which we do not have $\mathcal{J}\frac{a}{x} \vDash \phi$.

(b) For every value $a \in A$ such that $\mathcal{J}\frac{a}{x} \vDash \phi$ there is some $b \in A$ such that for all $c \in A$, if $\mathcal{J}\frac{b}{x}\frac{c}{y} \vDash \psi$ then $\mathcal{J}\frac{a}{x}\frac{b}{y} \vDash \phi$.

(c) There is some $a \in A$ that is in the image $f^A : A^2 \rightarrow A$ and some $b \in R^A$ such that $\mathcal{J}\frac{a}{x} \vDash \phi$ and $\mathcal{J}\frac{b}{x} \vDash \psi$.

(d) If there exists some $a \in A$ such that $\mathcal{J}\frac{a}{x} \vDash \phi$ then there is some $b \in A$ such that $\mathcal{J}\frac{b}{x} \vDash \psi$.

(e) If there is some $a \in A$ and some $b \in A$ such that $\mathcal{J}\frac{a}{x}\frac{b}{y} \vDash \phi$ then for every $c \in A$ we have $\mathcal{J}\frac{c}{x}\frac{c}{y} \vDash \phi$.

Solution. (a) (b) (c) (d) (e)