Quantum Information Processing Assignment 3

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I worked with Chelsea Komlo and Wilson Wu on this assignment.

Problem 1

Measuring the control qubit of a CNOT gate.

Solution. Lets start with the most general 2 qubit state.

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \tag{1}$$

Starting with the gate on the left, we first have to make a partial measurement of the first qubit. This collapses the state into one of two possibilities.

First qubit state	$\|0\rangle$	$\|1\rangle$
State after measurement	$\frac{1}{n}\|0\rangle \otimes \underbrace{(\alpha_{00}\|0\rangle + \alpha_{01}\|1\rangle)}_{\phi_0}$	$\frac{1}{n'}\|1\rangle \otimes \underbrace{(\alpha_{10}\|0\rangle + \alpha_{11}\|1\rangle)}_{\phi_1}$

Where $n = |a_{00}|^2 + |\alpha_{01}|^2$ and similarly with n' as factors for normalization. We can now feed these two states into a controlled U gate to see what comes out.

Measured $\|0\rangle$	Measured $\|1\rangle$
$\frac{1}{n}\begin{bmatrix}\mathbb{1} & 0\\ 0 & U\end{bmatrix}\begin{bmatrix}\phi_0\\ 0\end{bmatrix} = \frac{1}{n}\begin{bmatrix}\phi_0\\ 0\end{bmatrix}$	$\frac{1}{n'}\begin{bmatrix}\mathbb{1} & 0\\ 0 & U\end{bmatrix}\begin{bmatrix}0\\ \phi_1\end{bmatrix} = \frac{1}{n'}\begin{bmatrix}0\\ U\phi_1\end{bmatrix}$

Now we can switch our attention to the circuit on the right hand side. First we have to apply a controlled U to our general state defined in eq. (1).

$$\begin{bmatrix}\mathbb{1} & 0\\ 0 & U\end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ u_{11}\alpha_{10}+u_{12}\alpha_{11}\\ u_{21}\alpha_{10}+u_{22}\alpha_{11}\end{bmatrix}$$

Now with this state we must make a partial measurement of the first qubit. As before we will only measure $|0\rangle$ or $|1\rangle$ so we have two branches.

Measure $\|0\rangle$	Measure $\|1\rangle$
$\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ u_{11}\alpha_{10}+u_{12}\alpha_{11}\\ u_{21}\alpha_{10}+u_{22}\alpha_{11}\end{bmatrix} \to \frac{1}{n}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ 0\\ 0\end{bmatrix}$	$\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ u_{11}\alpha_{10}+u_{12}\alpha_{11}\\ u_{21}\alpha_{10}+u_{22}\alpha_{11}\end{bmatrix} \to \frac{1}{n'}\begin{bmatrix}0\\ 0\\ u_{11}\alpha_{10}+u_{12}\alpha_{11}\\ u_{21}\alpha_{10}+u_{22}\alpha_{11}\end{bmatrix}$

We now have the four states we need to conclude whether or not these circuits are the same. Measuring 0 in the first circuit led us to a state we called $\frac{1}{n}\begin{bmatrix}\phi_0\\ 0\end{bmatrix}$ which is shorthand for $\frac{1}{n}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ 0\\ 0\end{bmatrix}$. This is exactly what we got when first applying the controlled U and then making the measurement so that works out.

The case for measuring 1 then applying the controlled U gate got us $\frac{1}{n'}\begin{bmatrix}0\\ U\phi_1\end{bmatrix}$ which again was shorthand for $\frac{1}{n'}\begin{bmatrix}0\\ 0\\ u_{11}\alpha_{10}+u_{12}\alpha_{11}\\ u_{21}\alpha_{10}+u_{22}\alpha_{11}\end{bmatrix}$. This is what we got for the first circuit. At first I thought there might be a different normalization constant that needed to be added instead of just n', but because unitary operators are isometries (distance preserving maps), the length of the vector remains unchanged and hence still normalized.

Problem 2

Unitary between two triples of states with same inner products.

Solution. First assume the sets $\{\phi_i\}$ and $\{\psi_i\}$ are both linearly independent sets of vectors when considered on their own. Let us apply Gram-Schmidt orthonormalization to $\{\phi_i\}$ to obtain $\{\tilde{\phi}_i\}$ and similarly with $\{\psi_i\}$ to obtain $\{\tilde{\psi}_i\}$. We note that we now have two orthonormal bases for $\mathbb{C}^3$, and we know basis transformations are unitary maps. This means we can find a U such that $U|\tilde{\phi}_i\rangle = |\tilde{\psi}_i\rangle$.

Now we need to show $U|\phi_i\rangle = |\psi_i\rangle$. To do this lets look at how the $\{\tilde{\phi}_i\}$ were created.

$$|\tilde{\phi}_0\rangle = |\phi_0\rangle \qquad |\tilde{\psi}_0\rangle = |\psi_0\rangle$$

$$|\tilde{\phi}_1\rangle = |\phi_1\rangle - \frac{\langle\phi_1|\phi_0\rangle}{\langle\phi_0|\phi_0\rangle}|\phi_0\rangle \qquad |\tilde{\psi}_1\rangle = |\psi_1\rangle - \frac{\langle\psi_1|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}|\psi_0\rangle$$

$$|\tilde{\phi}_2\rangle = |\phi_2\rangle - \frac{\langle\phi_2|\tilde{\phi}_0\rangle}{\langle\tilde{\phi}_0|\tilde{\phi}_0\rangle}|\tilde{\phi}_0\rangle - \frac{\langle\phi_2|\tilde{\phi}_1\rangle}{\langle\tilde{\phi}_1|\tilde{\phi}_1\rangle}|\tilde{\phi}_1\rangle \qquad |\tilde{\psi}_2\rangle = |\psi_2\rangle - \frac{\langle\psi_2|\tilde{\psi}_0\rangle}{\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle}|\tilde{\psi}_0\rangle - \frac{\langle\psi_2|\tilde{\psi}_1\rangle}{\langle\tilde{\psi}_1|\tilde{\psi}_1\rangle}|\tilde{\psi}_1\rangle$$

It clear that $U|\tilde{\phi}_0\rangle = |\tilde{\psi}_0\rangle$ implies $U|\phi_0\rangle = |\psi_0\rangle$. Now let's apply U to $|\tilde{\phi}_1\rangle$.

$$\begin{aligned} U|\tilde{\phi}_1\rangle &= U|\phi_1\rangle - \langle\phi_1|\phi_0\rangle U|\phi_0\rangle \\ &= U|\phi_1\rangle - \langle\psi_1|\psi_0\rangle|\psi_0\rangle && \text{(bc inner prods equal, and } U|\phi_0\rangle = |\psi_0\rangle\text{.)} \\ &= |\psi_1\rangle - \langle\psi_1|\psi_0\rangle|\psi_0\rangle && \text{(RHS of } U|\tilde{\phi}_1\rangle = |\tilde{\psi}_1\rangle\text{.)} \end{aligned}$$

Where the last two lines show us $U|\phi_1\rangle = U|\psi_1\rangle$ as desired. A similar computation can be done with the third vector.

$$\begin{aligned} U|\tilde{\phi}_2\rangle &= U|\phi_2\rangle - \frac{\langle\phi_2|\tilde{\phi}_0\rangle}{\langle\tilde{\phi}_0|\tilde{\phi}_0\rangle}U|\tilde{\phi}_0\rangle - \frac{\langle\phi_2|\tilde{\phi}_1\rangle}{\langle\tilde{\phi}_1|\tilde{\phi}_1\rangle}U|\tilde{\phi}_1\rangle \\ &= U|\phi_2\rangle - \frac{\langle\phi_2|\tilde{\phi}_0\rangle}{\langle\tilde{\phi}_0|\tilde{\phi}_0\rangle}|\tilde{\psi}_0\rangle - \frac{\langle\phi_2|\tilde{\phi}_1\rangle}{\langle\tilde{\phi}_1|\tilde{\phi}_1\rangle}|\tilde{\psi}_1\rangle \\ &= |\psi_2\rangle - \frac{\langle\psi_2|\tilde{\psi}_0\rangle}{\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle}|\tilde{\psi}_0\rangle - \frac{\langle\psi_2|\tilde{\psi}_1\rangle}{\langle\tilde{\psi}_1|\tilde{\psi}_1\rangle}|\tilde{\psi}_1\rangle \end{aligned}$$

So as long as we can show the coefficients for the $|\tilde{\psi}_i\rangle$ terms are the same, we can conclude $U|\phi_2\rangle = |\psi_2\rangle$. The $|\tilde{\psi}_0\rangle$ is obviously the same because one can drop the tildes and use the fact that the vectors are normalized, and the inner products are pairwise equal. For the $|\tilde{\psi}_1\rangle$ term we need to compute some stuff.

$$\langle\phi_2|\tilde{\phi}_1\rangle = \overline{\alpha} - \overline{\alpha}^2$$

Here we've used $\alpha = \langle\phi_0|\phi_1\rangle$ and the fact that the inner product is conjugated when flipped. It's important to note here that this calculation is exactly the same for $\langle\psi_2|\tilde{\psi}_1\rangle$ because the inner products are equal (i.e. the indices match). The same logic and computation mean $\langle\tilde{\phi}_1|\tilde{\phi}_1\rangle = \langle\tilde{\psi}_1|\tilde{\psi}_1\rangle$. With this we conclude the problem solved for linearly independent sets of vectors.

Now if we assume the sets $\{\phi_i\}, \{\psi_i\}$ are linearly dependent, then we only have to show the first two vectors land in the right spots, because if $|\phi_2\rangle = \alpha|\phi_0\rangle + \beta|\phi_1\rangle$, then $U|\phi_2\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle = |\psi_2\rangle$ which is the only possibly option for $|\psi_2\rangle$ in order to preserve the inner products. The problem is trivial for all three vectors linearly dependent because a simple rotation will take a line to a line.