Quantum Information Processing Assignment 6

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I worked with Chelsea Komlo and Wilson Wu on this assignment.

Problem 1

Quantum Fourier transform.
(a) Prove that F_m is unitary.
(b) What computational basis state is $F_m^2 |a\rangle$?

Solution. (a) To show that F_m is unitary, we will show $(F_m)_i (F_m)_j^\dagger = \delta_{ij}$ where $(F_m)_i$ is the i-th row of F_m and $(F_m)_j^\dagger$ is the j-th column of $F_m^\dagger$.

$$\begin{aligned}
(F_m)_i (F_m)_j^\dagger &= \frac{1}{m}\begin{bmatrix} 1 & \omega^i & \omega^{2i} & \cdots & \omega^{(m-1)i} \end{bmatrix} \begin{bmatrix} 1 \\ \overline{\omega}^i \\ \overline{\omega}^{2i} \\ \vdots \\ \overline{\omega}^{(m-1)i} \end{bmatrix} \\
&= \frac{1}{m}\sum_{n=0}^{m-1} \omega^{ni}\overline{\omega}^{nj} \\
&= \frac{1}{m}\sum_{n=0}^{m-1} \omega^{ni}\omega^{-nj} \\
&= \frac{1}{m}\sum_{n=0}^{m-1} \omega^{n(i-j)}
\end{aligned}$$

If $i = j$, then we have $\frac{1}{m}\sum_{n=0}^{m-1} 1 = 1$. Now if $i \neq j$, say $i - j = a \in \mathbb{Z}$, then we have $\frac{1}{m}\sum_{n=0}^{m-1} \omega^{an} = 0$ by the sum of powers of roots of unity identities presented in lecture[1]. We thus conclude this product is δ_{ij} and hence equivalently we write $F_m F_m^\dagger = \mathbb{1}$.

(b) We'll first have to expand F_m^2, and we'll do it component-wise again.

$$(F_m^2)_{ik} = \sum_{j=1}^{m} (F_m)_{ij}(F_m)_{jk} = \sum_{j=1}^{m} \frac{1}{m}\omega^{(i-1)(j-1)}\omega^{(j-1)(k-1)} = \frac{1}{m}\sum_{j=0}^{m-1}\omega^{j(i+k-2)}$$

If $i + k - 2 \equiv 0 \mod m$, then $(F_m^2)_{ik} = 1$, and as above, if it's not 0, then $(F_m^2)_{ik} = 0$. This matrix isn't the identity, but it is full rank with only 1's and 0's as components. For a 4×4 we have the following.

$$F_4^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

[1] In lecture this was only presented for $a > 1$, but holds as well for negative a because ω^{-1} is also a primitive root of unity.

In general we have a 1 in the first position, and an $(m-1) \times (m-1)$ anti-diagonal matrix with 1s on the anti-diagonal with 0's everywhere else. This indeed maps computational basis states to computational basis states.

Problem 2

Superposition of eigenvectors in phase estimation algorithm.

Solution. Let $A = (F_{2^\ell}^* \otimes \mathbb{1})W(H^{\otimes n} \otimes \mathbb{1})$. Surely this in a linear operator. Thus we have the following.

$$\begin{aligned} A\left|0^\ell\right\rangle \otimes (\alpha_1\left|\psi_1\right\rangle + \alpha_2\left|\psi_2\right\rangle) &= \alpha_1 A\left|0^\ell\right\rangle \otimes \left|\psi_1\right\rangle + \alpha_2 A\left|0^\ell\right\rangle \otimes \left|\psi_2\right\rangle \\ &= \alpha_1\left|a\right\rangle \otimes \left|\psi_1\right\rangle + \alpha_2\left|b\right\rangle \otimes \left|\psi_2\right\rangle \end{aligned}$$

Written this way we can see when we measure the first ℓ qubits we will get a with probability $|\alpha_1|^2$ and b with probability $|\alpha_2|^2$.