# Quantum Information Processing Assignment 7

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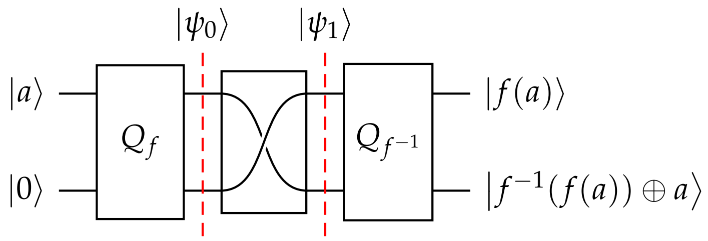
**Course:** QIC 710

I worked with Chelsea Komlo and Wilson Wu on this assignment.

**Problem 1**

Efficiently computing bijections.

**Solution.** The following circuit computes $U_f$.




Where we have the following intermediary states that follow directly from the definition of $Q_f$ and `SWAP`.

$$|\psi_0\rangle = |a\rangle |f(a)\rangle \qquad\qquad |\psi_1\rangle = |f(a)\rangle |a\rangle$$

Here the swap isn't really just one gate. In order to swap two qubits, it takes 3 `CNOT` gates, and we need to swap $n$ qubits in the top register with $n$ qubits in the bottom register. This is $\mathcal{O}(n)$ gates combined with one use of $Q_f$ and one use of $Q_{f^{-1}}$.

Lastly it's important to note that the bottom register is returned to $|0\rangle$ because $f^{-1}(f(a))$ is $a$ and $a \oplus a$ is 0.

**Problem 2**

Basic questions about density matrices.

(a) Show that, for any operator that is Hermitian, positive definite (i.e.,has no negative eigenvalues), and has trace 1, there is a probabilistic mixture of pure states whose density matrix is $\rho$.

(b) A density matrix $\rho$ corresponds to a pure state if and only if $\rho = |\psi\rangle\langle\psi|$. Show that any density matrix $\rho$ corresponds to a pure state if and only if $\operatorname{tr}(\rho^2) = 1$.

(c) Show that every $2 \times 2$ density matrix $\rho$ can be expressed as an equally weighted mixture of pure states.

**Solution.** (a) By the spectral theorem for Hermitian operators, we can write our Hermitian, positive definite operator as

$$A = \sum_i \lambda_i |i\rangle\langle i| .$$

This can be seen to be a density operator because each $\lambda_i$ is an eigenvalue of $A$, and by the Hermiticity of $A$ it is real and non-negative.

(b) Let $\rho = |\psi\rangle\langle\psi|$. Then $\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = |\psi\rangle\langle\psi| = \rho$. Thus $\operatorname{tr}(\rho^2) = \operatorname{tr}(\rho) = 1$.

Now take $\operatorname{tr}(\rho^2) = 1$. Let's first calculate $\rho^2$.

$$\begin{aligned}
\rho^2 &= \left(\sum_{i=1}^n p_i |i\rangle\langle i|\right)^2 = \sum_{i=1}^n p_i |i\rangle\langle i| \sum_{j=1}^n p_j |j\rangle\langle j| \\
&= \sum_{i,j=1}^n p_i p_j |i\rangle \langle i|j\rangle \langle j| \qquad \text{(using a basis where } \langle i|j\rangle = \delta_{ij}\text{)} \\
&= \sum_{i=1}^n p_i^2 |i\rangle\langle i|
\end{aligned}$$

Now taking the trace of $\rho^2$ we get a condition on the $p_i$'s.

$$\begin{aligned}
\operatorname{tr}\rho^2 &= \sum_{k=1}^n \langle k| \rho^2 |k\rangle \\
&= \sum_{k,i=1}^n \langle k| \left(p_i^2 |i\rangle\langle i|\right) |k\rangle \\
&= \sum_{i=1}^n p_i^2 = 1
\end{aligned}$$

So we now know $\sum p_i = 1 = \sum p_i^2$. By basic properties of real numbers, we know $p_i^2 \leq p_i$ when $p_i \in [0, 1]$ and equality only holding when $p_i \in \{0, 1\}$. Using this fact we can write

$$\sum_{i=1}^n p_i^2 \leq \sum_{i=1}^n p_i$$

where again equality only holds if $p_i \in \{0, 1\}$ for all $i$. The only way this can be true is if one of the $p_i$'s is 1 and all of the rest are 0. In that case our summation collapses to one term, and we are left with $\rho = |\psi_i\rangle\langle\psi_i|$ as desired.

(c) Let $\rho$ be an arbitrary $2 \times 2$ density matrix:

$$\rho = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| .$$

Define the following two vectors:

$$|\tilde{\psi}_0\rangle = \sqrt{\frac{p_0}{2}} |\psi_0\rangle + \sqrt{\frac{p_1}{2}} |\psi_1\rangle \qquad |\tilde{\psi}_1\rangle = \sqrt{\frac{p_0}{2}} |\psi_0\rangle - \sqrt{\frac{p_1}{2}} |\psi_1\rangle .$$

We'll now show that $\rho = \frac{1}{2} |\tilde{\psi}_0\rangle\langle\tilde{\psi}_0| + \frac{1}{2} |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1|$. For the page width's sake, let's calculate each term separately.

$$\begin{aligned}
|\tilde{\psi}_0\rangle\langle\tilde{\psi}_0| &= (\sqrt{p_0} |\psi_0\rangle + \sqrt{p_1} |\psi_1\rangle)(\sqrt{p_0} \langle\psi_0| + \sqrt{p_1} \langle\psi_1|) \\
&= p_0 |\psi_0\rangle\langle\psi_0| + \sqrt{p_0 p_1} |\psi_0\rangle\langle\psi_1| + \sqrt{p_0 p_1} |\psi_1\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1| \\
|\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| &= (\sqrt{p_0} |\psi_0\rangle - \sqrt{p_1} |\psi_1\rangle)(\sqrt{p_0} \langle\psi_0| - \sqrt{p_1} \langle\psi_1|) \\
&= p_0 |\psi_0\rangle\langle\psi_0| - \sqrt{p_0 p_1} |\psi_0\rangle\langle\psi_1| - \sqrt{p_0 p_1} |\psi_1\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1|
\end{aligned}$$

Thus putting those together we have

$$\frac{1}{2} |\tilde{\psi}_0\rangle\langle\tilde{\psi}_0| + \frac{1}{2} |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| = p_0 |\psi_0\rangle\langle\psi_0| + p_1 |\psi_1\rangle\langle\psi_1|$$

Thus we conclude it's possible to write an arbitrary density matrix as an equally weighted mixture of pure states.

**Problem 3**

The density matrix depends on what you know.

(a) What is the density matrix of the state from Bob's perspective?

(b) What's Alice's density matrix for Bob's state assuming that her initial state was $|0\rangle$? What's Alice's density matrix for Bob's state assuming that her initial state was $|+\rangle$?

(c) What is the density matrix of the state from Bob's perspective? Is it the same matrix as in part (a)?

(d) What's Alice's density matrix for Bob's state assuming that her initial state was $|\psi_0\rangle$? What's Alice's density matrix for Bob's state assuming that her initial state was $|\psi_1\rangle$?

**Solution.** (a) From Bob's perspective, he knows that there is a 50% chance he's getting $|0\rangle$ and a 50% chance he's getting $|+\rangle$. With that information we can write down the following density matrix.

$$\rho_{\text{Bob}} = \sum_i p_i |i\rangle\langle i| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

(b) Assuming Alice's initial state was $|0\rangle$, then she knows exactly which state she sent to Bob and hence she knows what state he measured: $\rho_{\text{Alice},|0\rangle} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. When she sends the $|+\rangle$, Bob will measure $|0\rangle$ and $|1\rangle$ with equal probabilities. Thus the off diagonal terms vanish and we are left with

$$\rho_{\text{Alice},|+\rangle} = \sum_i p_i |i\rangle\langle i| = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(c) From Bob's perspective, he's getting $|\psi_0\rangle$ with probability $\cos^2(\pi/8)$ and $|\psi_1\rangle$ with probability $\sin^2(\pi/8)$. Thus from his perspective he has the following density matrix.

$$\begin{aligned}
\rho_{\text{Bob}} &= \cos^2(\pi/8)\,|\psi_0\rangle\langle\psi_0| + \sin^2(\pi/8)\,|\psi_1\rangle\langle\psi_1| \\
&= \begin{bmatrix} \cos^4(\frac{\pi}{8}) & \cos^3(\frac{\pi}{8})\sin(\frac{\pi}{8}) \\ \cos^3(\frac{\pi}{8})\sin(\frac{\pi}{8}) & \cos^2(\frac{\pi}{8})\sin^2(\frac{\pi}{8}) \end{bmatrix} + \begin{bmatrix} \sin^4(\frac{\pi}{8}) & -\cos(\frac{\pi}{8})\sin^3(\frac{\pi}{8}) \\ -\cos(\frac{\pi}{8})\sin^3(\frac{\pi}{8}) & \cos^2(\frac{\pi}{8})\sin^2(\frac{\pi}{8}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad \text{(by judicious use of wolfram alpha)}
\end{aligned}$$

Interestingly, this is the same as in (a)!

(d) Start with Alice sending $|\psi_0\rangle$. Similar to what we saw in part (b), Alice knows that once Bob measure the state will be in state $|0\rangle$ with probability $\cos^2(\frac{\pi}{8})$ and in state $|1\rangle$ with probability $\sin^2(\frac{\pi}{8})$.

$$\rho_{\text{Alice},|\psi_0\rangle} = \begin{bmatrix} \frac{1}{4}(2+\sqrt{2}) & 0 \\ 0 & \frac{1}{4}(2-\sqrt{2}) \end{bmatrix}$$

A similar computation can be done if Alice sends $|\psi_1\rangle$.

$$\rho_{\text{Alice},|\psi_1\rangle} = \begin{bmatrix} \frac{1}{4}(2-\sqrt{2}) & 0 \\ 0 & \frac{1}{4}(2+\sqrt{2}) \end{bmatrix}$$