Quantum Information Processing Assignment 10

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I worked with Chelsea Komlo and Wilson Wu on this assignment.

Problem 1

Prove that $R_1 R_2$ is a rotation by angle $2(\theta_1 - \theta_2)$.

Solution. Here we use $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ to denote the two states we will use to define R_1 and R_2 respectively. This is only done because it's easier to typeset than $|\psi_{\theta_1}\rangle$. The first thing we'll need to do calculate $|\psi_\alpha\rangle\langle\psi_\alpha|$.

$$\begin{aligned}
|\psi_\alpha\rangle\langle\psi_\alpha| &= (\cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle)(\cos\alpha\,\langle 0| + \sin\alpha\,\langle 1|) \\
&= \cos^2\alpha\,|0\rangle\langle 0| + \sin^2\alpha\,|1\rangle\langle 1| + \sin\alpha\cos\alpha\,|1\rangle\langle 0| + \cos\alpha\sin\alpha\,|0\rangle\langle 1| \\
|\psi_\alpha^\perp\rangle\langle\psi_\alpha^\perp| &= \sin^2|0\rangle\langle 0| + \cos^2\alpha\,|1\rangle\langle 1| - \sin\alpha\cos\alpha\,|0\rangle\langle 1| - \sin\alpha\cos\alpha\,|1\rangle\langle 0|
\end{aligned}$$

With these we can now calculate R_1.

$$\begin{aligned}
R_1 &= |\psi_\alpha\rangle\langle\psi_\alpha| - |\psi_\alpha^\perp\rangle\langle\psi_\alpha^\perp| \\
&= \cos 2\alpha\,|0\rangle\langle 0| - \cos 2\alpha\,|1\rangle\langle 1| + \sin 2\alpha\,|1\rangle\langle 0| + \sin 2\alpha\,|0\rangle\langle 1|
\end{aligned}$$

where I've used the fact that $\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$. Now R_2 should have the exact same form, but with β in place of α. We're now ready to calculate $R_1 R_2$.

$$\begin{aligned}
R_1 R_2 &= (\cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta)\,|0\rangle\langle 0| \\
&\quad + (\cos 2\alpha\sin 2\beta - \sin 2\alpha\cos 2\beta)\,|0\rangle\langle 1| \\
&\quad + (\cos 2\beta\sin 2\alpha - \cos 2\alpha\sin 2\beta)\,|1\rangle\langle 0| \\
&\quad + (\cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta)\,|1\rangle\langle 1| \\
&= \cos(2(a-b))\,|0\rangle\langle 0| - \sin(2(a-b))\,|0\rangle\langle 1| \\
&\quad + \sin(2(a-b))\,|1\rangle\langle 0| + \cos(2(a-b))\,|1\rangle\langle 1|
\end{aligned}$$

Now using the fact that $|0\rangle$ and $|1\rangle$ represent the standard basis, we can expand $|0\rangle\langle 0|$ into matrices to see that $R_1 R_2$ takes the following form.

$$R_1 R_2 = \begin{bmatrix} \cos(2(a-b)) & -\sin(2(a-b)) \\ \sin(2(a-b)) & \cos(2(a-b)) \end{bmatrix}$$

This is the same form as an arbitrary rotation R_θ in two dimensions.

Problem 2

(a) Show that, for any classical algorithm, the number of f-queries required to solve this problem exactly is exponential in n.

(b) Show that there is a quantum algorithm that makes one single f-query and is guaranteed to find an $x \in \{0,1\}^n$ such that $f(x) = 1$. (Hint: consider what a single iteration fo Grover's algorithm does.)

Solution. (a) It's possible for a classical algorithm to query $\frac{3}{4}$ of the entire space $\{0,1\}^n$ and not find a satisfying assignment. It's only once it queries in $\frac{3}{4} + 1$ places can we guarantee that it finds a satisfying assignment. Thus we conclude the worst case scenario requires $\sim \frac{3}{4}2^n$ queries.

(b) Let's first take a look at our state $H|0^n\rangle$.

$$H|0^n\rangle = \sqrt{\frac{s_0}{2^n}}|A_0\rangle + \sqrt{\frac{s_1}{2^n}}|A_1\rangle = \frac{\sqrt{3}}{4}|A_0\rangle + \frac{1}{2}|A_1\rangle$$

Here we've used the fact that $s_0 = \frac{3}{4}2^n$ and $s_1 = \frac{1}{4}2^n$. Now note that the angle this vector makes on the $|A_0\rangle, |A_1\rangle$ plane is $\theta = \frac{\pi}{6}$. Now when we perform a single iteration of Grover's algorithm we will be rotated to a vector with angle 3θ with respect to the $|A_0\rangle$ axis. Lucky for us $3\theta = \frac{\pi}{2}$. Thus we've landed right on $|A_1\rangle$, and hence we've only made 1 query to f, and gotten our state vector in the right place. We can now measure it to obtain one of the (possibly) many values for which $f(x) = 1$.