Advanced Quantum Theory Homework 3

Name: Nate Stemen (20906566)
Email: nate.stemen@uwaterloo.ca
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Exercise 15

Show that for any arbitrary choice of complex numbers r, s the matrix-valued functions $\hat{x}(t)$ and $\hat{p}(t)$ defined through Eqs. 3.51, 3.52 obey the equations of motion at all time.

Solution. We first have to calculate the equation of motion besides the given $\hat{p} = m\dot{\hat{x}}(t)$.

$$\begin{aligned}
\dot{\hat{p}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{p} &= \frac{1}{\mathrm{i}\hbar}\left[\hat{p}, \widehat{H}\right] \\
&= \frac{1}{\mathrm{i}\hbar}\left[\hat{p}, \frac{m\omega^2}{2}\hat{x}^2\right] \\
&= \frac{m\omega^2}{2\mathrm{i}\hbar}\left[\hat{p}, \hat{x}^2\right] \\
&= \frac{m\omega^2}{2\mathrm{i}\hbar}(-2\mathrm{i}\hbar\hat{x}) \\
&= -m\omega^2\hat{x} = -k\hat{x}
\end{aligned}$$

Just like we remember from classical mechanics. Now we need to ensure our ansatz works for this equation of motion.

$$\begin{aligned}
\dot{\hat{p}}(t) = m\ddot{\hat{x}} &= m\left(\ddot{\xi}a + \ddot{\bar{\xi}}a^\dagger\right) \\
&= m\left[\left(-r\omega^2\sin(\omega t) - s\omega^2\cos(\omega t)\right)a + \left(-\bar{r}\omega^2\sin(\omega t) - \bar{s}\omega^2\cos(\omega t)\right)a^\dagger\right] \\
&= -m\omega^2\left[(-r\sin(\omega t) - s\cos(\omega t))a + (-\bar{r}\sin(\omega t) - \bar{s}\cos(\omega t))a^\dagger\right] \\
&= -m\omega^2\left[\xi(t)a + \bar{\xi}(t)a^\dagger\right] \\
&= -m\omega\hat{x}(t) = -k\hat{x}(t)
\end{aligned}$$

Sweet.

Exercise 16

Show that, again for any arbitrary choice of complex numbers r, s, the matrix-valued functions $\hat{x}(t)$ and $\hat{p}(t)$ defined through Eqs. 3.51, 3.52 obey the hermiticity conditions at all time.

Solution. Note: we are now using a star/asterisk to denote complex conjugate whereas in the previous question we used an overline. Sorry for any confusion.

$$\begin{aligned} \hat{x}^\dagger &= \left(\xi(t)a + \xi^*(t)a^\dagger\right)^\dagger & \hat{p}^\dagger &= m^*\left(\dot{\xi}(t)a + \dot{\xi}^*(t)a^\dagger\right)^\dagger \\ &= \xi^*(t)a^\dagger + \xi(t)a = \hat{x} & &= m\left(\dot{\xi}^*(t)a^\dagger + \dot{\xi}(t)a\right) = \hat{p} \end{aligned}$$

Here we've not actually had to use the ansatz $\xi(t)$, and hence we have shown that the position and momentum operators are hermitian for all r, s.

Exercise 17

Find the equation that the complex numbers r, s have to obey so that the matrix-valued functions $\hat{x}(t)$ and $\hat{p}(t)$ defined through Eqs. 3.51, 3.52 obey the canonical commutation relations at all time. This equation for r, s is called the Wronskian condition and it has many solutions. Give an example of a pair of complex numbers r, s that obey the Wronskian condition and write down $\hat{x}(t)$ explicitly with these values for r, s filled in.

Solution. This question is a computational doozie... We need to ensure $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$.

$$\begin{aligned}
[\hat{x}, \hat{p}] &= m\left(\xi a + \xi^* a^\dagger\right)\left(\dot{\xi} a + \dot{\xi}^* a^\dagger\right) - m\left(\dot{\xi} a + \dot{\xi}^* a^\dagger\right)\left(\xi a + \xi^* a^\dagger\right) \\
&= m\left[\xi(t)\dot{\xi}^*(t) - \dot{\xi}(t)\xi^*(t)\right] \\
&= m\omega(sr^* - rs^*) \\
&= m\omega(a_1 + \mathrm{i}a_1)(b_1 - \mathrm{i}b_2) - (b_1 + \mathrm{i}b_2)(a_1 - \mathrm{i}a_2) \\
\mathrm{i}\hbar &= 2m\omega\mathrm{i}[a_2 b_1 - a_1 b_2] \\
a_2 b_1 - a_1 b_2 &= \frac{\hbar}{2m\omega}
\end{aligned}$$

From here we can let $s = 0 + \mathrm{i}\frac{\hbar}{2m\omega}$ and $r = 1 + \mathrm{i}666$ and we can write down our position operator.

$$\begin{aligned}
\hat{x}(t) = &\left((1 + \mathrm{i}666)\sin(\omega t) + \mathrm{i}\frac{\hbar}{2m\omega}\cos(\omega t)\right)a \\
&+ \left((1 - \mathrm{i}666)\sin(\omega t) - \mathrm{i}\frac{\hbar}{2m\omega}\cos(\omega t)\right)a^\dagger
\end{aligned}$$

Exercise 18

Use Eqs. 3.51, 3.52 to express the Hamiltonian in terms of functions and the operators $a, a^\dagger$. There should be terms proportional to a^2, to $(a^\dagger)^2$, $aa^\dagger$ and $a^\dagger a$.

Solution.

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \\ &= \left[\frac{m}{2}\dot{\xi}^2 + \frac{m\omega^2}{2}\xi^2\right]a^2 + \left[\frac{m}{2}\dot{\xi}^{*2} + \frac{m\omega^2}{2}\xi^{*2}\right]a^{\dagger 2} \\ &\quad + \left[\frac{m}{2}\dot{\xi}\dot{\xi}^* + \frac{m\omega^2}{2}\xi\xi^*\right]aa^\dagger + \left[\frac{m}{2}\dot{\xi}\dot{\xi}^* + \frac{m\omega^2}{2}\xi\xi^*\right]a^\dagger a\end{aligned}$$

Exercise 19

It turns out that it is possible to choose the coefficients r and s so that the terms in the Hamiltonian which are proportional to a^2 and $(a^\dagger)^2$ drop out. Find the condition which the equation that r and s have to obey for this to happen. Choose a pair of complex numbers r, s such that the Hamiltonian simplifies this way, and of course such that the Wronskian condition is obeyed. Write down $\hat{H}(t)$ as an explicit matrix for this choice of r, s. It should be a diagonal matrix.

Solution. We can expand the a^2 and $a^{\dagger 2}$ to get the following condtions on r and s.

$$\frac{m}{2}\dot{\xi}^2 + \frac{m\omega^2}{2}\xi^2 = \frac{m\omega^2}{2}\left(r^2 + s^2\right) \implies r^2 = -s^2$$
$$\frac{m}{2}\dot{\xi}^{*2} + \frac{m\omega^2}{2}\xi^{*2} = \frac{m\omega^2}{2}\left(r^{*2} + s^{*2}\right) \implies r^{*2} = -s^{*2}$$

Turns out these equations are equivalent (no duh, you might say). So we have to satisfy the following three conditions simultaneously where $r = a_1 + \mathrm{i}a_2$ and $s = b_1 + \mathrm{i}b_2$.

$$a_2 b_1 - a_1 b_2 = \frac{\hbar}{2m\omega}$$
$$b_1^2 - b_2^2 = a_2^2 - a_1^2$$
$$b_1 b_2 = -a_1 a_2$$

The following values are a quadruple that satisfy the above equations.

$$a_1 = \left(\frac{-\hbar}{4m\omega}\right)^{\frac{1}{2}} \qquad b_1 = -\left(\frac{-\hbar}{4m\omega}\right)^{\frac{1}{2}}$$
$$a_2 = \left(\frac{-\hbar}{4m\omega}\right)^{\frac{1}{2}} \qquad b_2 = \left(\frac{-\hbar}{4m\omega}\right)^{\frac{1}{2}}$$

The Hamiltonian would then read

$$\widehat{H} = -\frac{\hbar\omega}{2}\begin{bmatrix} 1 & & & & \\ & 3 & & & \\ & & 5 & & \\ & & & 7 & \\ & & & & \ddots \end{bmatrix}$$

where the off diagonal elements are 0.

Exercise 20

Give a counter example for Eq. 3.67. To this end, write out Eq. 3.67 explicitly, i.e., in matrix form, for the case $\hat{f}(\hat{x}(t), \hat{p}(t)) = \hat{x}^2$. Then choose a suitable normalized ψ so that Eq. 3.67 is seen to be violated. (It is not difficult to find such a ψ, almost every one will do.)

Solution. If $\hat{f}(\hat{x}, \hat{p}) = \hat{x}^2$, then the expectation $\bar{f}$ is as follows.

$$\bar{f} = \sum_{n,m=1}^{\infty} \psi_n^* \left[\hat{x}^2\right]_{n,m} \psi_m$$

On the other hand if we look at $f(\overline{x}, \overline{p})$ then we have the following.

$$f(\overline{x}, \overline{p}) = (\overline{x})^2 = \left[\sum_{n,m=1}^{\infty} \psi_n^* \hat{x}_{n,m} \psi_m\right]^2$$

Now let $\psi = \begin{bmatrix} 1 & 0 & \cdots \end{bmatrix}^\intercal$ (all zeros except for the first element). Then we have $\bar{f} = \left[\hat{x}^2\right]_{0,0}$ and $f(\overline{x}, \overline{p}) = (\hat{x}_{0,0})^2$. Because $[\hat{x}^2]_{0,0} = x_{0k}x_{k0}$ where the sum over k is implied, this is not in general equal to $x_{0,0}^2$. Even given the fact that $\hat{x}$ is hermitian, and hence $x_{0k} = x_{k0}^*$ so $[\hat{x}^2]_{0,0} = x_{0k}^2$ it's still not generally true.

Exercise 21

Verify that ψ of Eq. 3.61 is normalized. For this choice of ψ, calculate explicitly the expectation values $\bar{x}(t)$, $\bar{p}(t)$ as well as the uncertainties in those predictions, i.e., the standard deviations $\Delta x(t)$ and$\Delta p(t)$ for the free particle. Your results should show that neither the position nor the momentum are predicted with certainty at any time, not even at the initial time t_0. The fact that $\Delta x(t)$ grows in time expresses that a momentum uncertainty over time leads to increasing position uncertainty. $\Delta p(t)$ remains constant in time, expressing that the momentum of a free particle, no matterwhat value it has, remains unchanged.

Solution. First, the fact that ψ is normalized:

$$\frac{1}{5}\begin{bmatrix}4 & -3\mathrm{i} & 0 & \cdots\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 \\ 3\mathrm{i} \\ 0 \\ \vdots\end{bmatrix} = \frac{1}{25}(16+9) = 1$$

Exercise 22

Spell out the step of the second equality in Eq. 3.68.

Solution. Pretty much all of the of the manipulations that follow are because of the fact that the expectation value of a constant is just that constant. So $\overline{Q\overline{Q}} = \overline{Q}^2$.

$$\begin{aligned}
(\Delta Q)^2 &= \overline{(Q - \overline{Q})^2} \\
&= \overline{Q^2 + \overline{Q}^2 - 2Q\overline{Q}} \\
&= \overline{Q^2} + \overline{\overline{Q}^2} - 2\overline{Q\overline{Q}} \\
&= \overline{Q^2} + \overline{Q}^2 - 2\overline{Q}\overline{Q} \\
&= \overline{Q^2} - \overline{Q}^2
\end{aligned}$$

Exercise 23

Verify that $\mathcal{H}^*$ is a complex vector space.

Solution. Define the addition of "bra" vectors as follows.

$$(\langle\phi| + \langle\psi|)\,|\xi\rangle \mapsto \langle\phi|\xi\rangle + \langle\psi|\xi\rangle$$

With this defined, we have to show it forms an abelian group. First, closure. For $\langle\phi|\,,\langle\psi| \in \mathcal{H}$ there sum is also in $\mathcal{H}$ because the sum of two complex numbers is again a complex number. Secon, associativity. This follows from the fact that the addition of complex numbers is associative. Third, identity. Let the identity element be $\langle 0|$ which is the linear functional that sends everything to 0. With this it's clear $\langle\phi| + \langle 0| = \langle 0| + \langle\phi| = \langle\phi|$. Fourth, inverses. The inverse of $\langle\phi|$ will be defined by $-\langle\phi|$. We then have $(\langle\phi| + (-\langle\phi|))\,|\xi\rangle = \langle\phi|\xi\rangle - \langle\phi|\xi\rangle = 0$ which is equivalent to the action of the 0 bra vector. Lastly, commutativity. This follows easily from the commutativity of complex number addition.

Now we need to define scalar multiplication. Define the action as follows.

$$(\alpha\,\langle\phi|)\,|\psi\rangle = \alpha\,\langle\phi|\psi\rangle$$

Now we need to verify some properties about the interaction of scalar multiplication and bra vector addition.

$$\begin{aligned}
((\alpha+\beta)\,\langle\phi|)\,|\psi\rangle &= (\alpha+\beta)\,\langle\phi|\psi\rangle \\
&= \alpha\,\langle\phi|\psi\rangle + \beta\,\langle\phi|\psi\rangle \\
&= (\alpha\,\langle\phi|)\,|\psi\rangle + (\beta\,\langle\phi|)\,|\psi\rangle \\
(\alpha(\langle\phi| + \langle\psi|))\,|\xi\rangle &= \alpha\,\langle\phi|\xi\rangle + \alpha\,\langle\psi|\xi\rangle \\
((\alpha\beta)\,\langle\phi|)\,|\psi\rangle &= (\alpha\beta)\,\langle\phi|\psi\rangle \\
&= \alpha(\beta\,\langle\phi|\psi\rangle) \\
(1\,\langle\phi|)\,|\xi\rangle &= 1\,\langle\phi|\xi\rangle \\
&= \langle\phi|\xi\rangle
\end{aligned}$$

With all of these properties satisfied we can conclude that $\mathcal{H}^*$ is indeed a vector space.