Advanced Quantum Theory Homework 6

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Exercise 1

Assume that $\hat{f}(t)$ is any observable which does not explicitly depend on time (i.e., which is a polynomial or a well-behaved power series in the position and momentum operators with constant coefficients). Show that the time evolution of any such $\hat{f}(t)$ is given by:

$$\hat{f}(t) = \hat{U}^\dagger(t)\hat{f}(t_0)\hat{U}(t).$$

Solution. Let $P_n(\hat{x}, \hat{p})$ denote a single polynomial term of degree n in the $\hat{x}$ and $\hat{p}$'s. For example $P_n(\hat{x}, \hat{p})$ could be $\hat{x}^n$, or $\hat{x}^{n/2}\hat{p}^{n/2}$ or $\hat{x}\hat{p}\hat{x}\cdots\hat{x}\hat{p}$ where $\hat{x}\hat{p}$ is repeated $n/2$ times. With this notation we can then write our function $\hat{f}$ as

$$\hat{f}(t) = \sum_n \alpha_n P_n(\hat{x}(t), \hat{p}(t))$$

Now before showing the time evolution of $\hat{f}$ is given as above we will show the time evolution of the individual terms is given by

$$P_n(\hat{x}(t), \hat{p}(t)) = \hat{U}^\dagger(t) P_n(\hat{x}(t_0), \hat{p}(t_0))\hat{U}(t). \tag{1}$$

We'll follow the tried and tested method of inserting identities (but really $\hat{U}^\dagger\hat{U}$) between every term. This is the part I don't know how to show in symbols. It makes complete sense to me, and all of the examples I provided above work out perfectly, but I don't know how to write a general term of P_n out so that I can insert identities between the terms. Maybe something like $P_n = \prod_{p_i} \hat{v}^{p_i}$ where $\hat{v} \in \{\hat{x}, \hat{p}\}$ and $\sum_i p_i = n$. Then I think that would work because the $\hat{x}$ and the $\hat{p}$'s behave the same way when doing this. So I'll take that **??** works. Now let's conjugate $\hat{f}(t_0)$.

$$\begin{aligned}
\hat{U}^\dagger(t)\hat{f}(t_0)\hat{U}(t) &= \hat{U}^\dagger(t)\left[\sum_n \alpha_n P_n(\hat{x}(t_0), \hat{p}(t_0))\right]\hat{U}(t) \\
&= \sum_n \alpha_n \hat{U}^\dagger(t) P_n(\hat{x}(t_0), \hat{p}(t_0))\hat{U}(t) \\
&= \sum_n \alpha_n P_n(\hat{x}(t), \hat{p}(t)) \\
&= \hat{f}(t)
\end{aligned}$$

Exercise 2

Bonus question:

Solution. Sorry, super busy this week so no extra time unfortunately ☹.

Exercise 3

(a) Use the time evolution operator to prove that the canonical commutation relations are conserved, i.e., that, for example, $[\hat{x}(t_0), \hat{p}(t_0)] = \mathrm{i}\hbar$ implies $[\hat{x}(t), \hat{p}(t)] = \mathrm{i}\hbar$ for all t.

(b) Consider the possibility that (due to quantum gravity effects) at some time t_0 the xp commutation relations take the form $[\hat{x}(t_0), \hat{p}(t_0)] = \mathrm{i}\hbar(1 + \beta\hat{p}(t_0)^2)$ (where β is a small positive constant). Assume that the Hamiltonian is self-adjoint, i.e., that the time evolution operator is still unitary. Will these commutation relations be conserved under the time evolution?

Solution. **??** Let's start by conjugating the commutator at the initial time t_0.

$$\begin{aligned}
\mathrm{i}\hbar &= \hat{U}^\dagger(t)[\hat{x}(t_0), \hat{p}(t_0)]\hat{U}(t) \\
&= \hat{U}^\dagger(t)\hat{x}(t_0)\hat{p}(t_0)\hat{U}(t) - \hat{U}^\dagger(t)\hat{p}(t_0)\hat{x}(t_0)\hat{U}(t) \\
&= \hat{U}^\dagger(t)\hat{x}(t_0)\hat{U}(t)\hat{U}^\dagger(t)\hat{p}(t_0)\hat{U}(t) - \hat{U}^\dagger(t)\hat{p}(t_0)\hat{U}(t)\hat{U}^\dagger(t)\hat{x}(t_0)\hat{U}(t) \\
&= \hat{x}(t)\hat{p}(t) - \hat{p}(t)\hat{x}(t) \\
&= [\hat{x}(t), \hat{p}(t)]
\end{aligned}$$

Feeling thankful for copy and paste right now, my best buds.

?? In part **??** we demonstrated that when we conjugate the commutator of $\hat{x}(t_0)$ and $\hat{p}(t_0)$ it evolves to the commutator of $\hat{x}(t)$ and $\hat{p}(t)$. So to answer this part of the question we will show $\mathrm{i}\hbar(1 + \beta\hat{p}(t_0)^2) \xmapsto{\hat{U}^\dagger \bigstar \hat{U}} \mathrm{i}\hbar(1 + \beta\hat{p}(t)^2)$ where I've used $\bigstar$ to denote the thing being conjugated.

$$\begin{aligned}
\hat{U}^\dagger(t)\mathrm{i}\hbar(1 + \beta\hat{p}(t_0)^2)\hat{U}(t) &= \mathrm{i}\hbar\left[\hat{U}^\dagger(t)\hat{U}(t) + \beta\hat{U}^\dagger(t)\hat{p}(t_0)\hat{p}(t_0)\hat{U}(t)\right] \\
&= \mathrm{i}\hbar\left[\mathbb{1} + \beta\hat{U}^\dagger(t)\hat{p}(t_0)\hat{U}(t)\hat{U}^\dagger(t)\hat{p}(t_0)\hat{U}(t)\right] \\
&= \mathrm{i}\hbar\left[\mathbb{1} + \beta\hat{p}(t)^2\right]
\end{aligned}$$

Thus we conclude even a modified commutation relation like the above is conserved with unitary time evolution. Quantum gravity solved.

Exercise 4

Consider a system with a Hamiltonian that has no explicit time dependence. Assume that we prepare the system in a state so that its energy at the initial time t_0 is known precisely.

(a) Show that the energy of the system will stay sharp, i.e., without uncertainty, at that value.

(b) Consider now the specific example of a harmonic oscillator system. Its positions and momenta evolve according to Eqs.7.26. Given the time-energy uncertainty relations, what more can you conclude for the time-evolution of $\overline{x}(t)$ and $\overline{p}(t)$ if the system is in a state with vanishing uncertainty in the energy?

Solution. **??** In the next question we show that when $\hat{H}$ has no explicit time dependence, then it commutes with the time evolution operator $\hat{U}(t)$. Using this fact we have

$$\hat{H}(t) = \hat{U}^\dagger(t)\hat{H}(t_0)\hat{U}(t) = \hat{H}(t_0)\hat{U}^\dagger(t)\hat{U}(t) = \hat{H}(t_0).$$

Thus if $\left(\Delta\hat{H}(t)\right)^2 = \left\langle\hat{H}^2(t)\right\rangle - \left\langle\hat{H}(t)\right\rangle^2$ then inserting the above clearly shows the $\left(\Delta\hat{H}(t_0)\right)^2$ is the same.

?? Given $\Delta E = 0$, then Δt must go infinity, and hence the expectation values $\overline{x}(t)$ and $\overline{p}(t)$ will follow the same patterns for all time, until something disturbs the system. Dang disturbances.

Exercise 5

Eq.8.37 shows that, in general, $\hat{H} \neq \hat{H}_S$ because in general the Heisenberg Hamiltonian does not commute with the time evolution operator. And this is because time-dependent Heisenberg Hamiltonians generally don't even commute with themselves at different times. Show that if the Heisenberg Hamiltonian $\hat{H}$ does not explicitly depend on time (i.e., if it is a polynomial in the $\hat{x}$ and $\hat{p}$ with time-independent coefficients, i.e., if we do not introduce an explicit time-dependence manually) then it coincides with the Schrödinger Hamiltonian.

Solution. When $\hat{H}$ has no time dependence then $\hat{U}$ is defined as

$$\hat{U}(t) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t - t_0}{\mathrm{i}\hbar} \right)^n \hat{H}^n.$$

We'll now show that this time evolution operator commutes with the Heisenberg Hamiltonian.

$$\hat{U}(t)\hat{H} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t - t_0}{\mathrm{i}\hbar} \right)^n \hat{H}^{n+1} = \hat{H} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t - t_0}{\mathrm{i}\hbar} \right)^n \hat{H}^n = \hat{H}\hat{U}(t)$$

With this let's take a look at what the Schrödinger Hamiltonian looks like.

$$\hat{H}_S := \hat{U}(t)\hat{H}\hat{U}^\dagger(t) = \hat{H}\hat{U}(t)\hat{U}^\dagger(t) = \hat{H}$$

Thus we've shown when either the Heisenberg/Schrödinger Hamiltonian has no explicit time dependence, then they are equal.

Exercise 6

Assuming that $\hat{f}$ is an observable that has no explicit time dependence (i.e., that depends on time only through the operators $\hat{x}(t)$ and $\hat{p}(t)$), show that the following equation holds true in the Schrödinger picture and in the Heisenberg picture:

$$i\hbar \frac{d}{dt} \langle \psi | \hat{f} | \psi \rangle = \langle \psi | [\hat{f}, \hat{H}] | \psi \rangle .$$

Solution. Let's first do the Heisenberg picture where the states $|\psi\rangle$ are frozen in time and hence have no time dependence.

$$\begin{aligned}
i\hbar \frac{d}{dt} \langle \psi | \hat{f} | \psi \rangle &= i\hbar \langle \psi | \dot{\hat{f}} | \psi \rangle \\
&= i\hbar \frac{d}{dt} \langle \psi | \{\hat{f}, \hat{H}\} + \partial_t f | \psi \rangle && \text{(Hamilton's equation)} \\
&= \frac{d}{dt} \langle \psi | [\hat{f}, \hat{H}] | \psi \rangle && \text{(by } \{\hat{f}, \hat{H}\} = \tfrac{1}{i\hbar}[\hat{f}, \hat{H}] \text{ and } \partial_t f = 0)
\end{aligned}$$

Now the Schrödinger picture. Here we will use $i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t) |\psi(t)\rangle$ in the derivation.

$$\begin{aligned}
i\hbar \frac{d}{dt} \langle \psi | \hat{f} | \psi \rangle &= i\hbar \langle \dot{\psi}(t) | \hat{f} | \psi(t) \rangle + i\hbar \langle \psi(t) | \hat{f} | \dot{\psi}(t) \rangle \\
&= - \langle \psi(t) | \hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t)\hat{f} | \psi(t) \rangle + \langle \psi(t) | \hat{f}\hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t) | \psi(t) \rangle \\
&= \langle \psi(t) | \hat{f}\hat{H}_S(t) - \hat{H}_S(t)\hat{f} | \psi(t) \rangle \\
&= \langle \psi | [\hat{f}, \hat{H}_S] | \psi \rangle
\end{aligned}$$

Thus we've shown the equation to hold true in both the Heisenberg and the Schrödinger picture. Cool stuff.

Exercise 7

Show that $\hat{U}'(t)$ is unitary.

Solution.

$$\left[\hat{U}'(t)\right]^{\dagger}\hat{U}'(t) = \left[\hat{U}^{(e)\dagger}(t)\hat{U}(t)\right]^{\dagger}\hat{U}^{(e)\dagger}(t)\hat{U}(t) = \hat{U}^{\dagger}(t)\hat{U}^{(e)}(t)\hat{U}^{(e)\dagger}(t)\hat{U}(t) = \mathbb{1}$$

Where we've used the fact that $\hat{U}^{(e)}(t)\hat{U}^{(e)\dagger}(t) = \mathbb{1}$ along with $\hat{U}^{\dagger}(t)\hat{U}(t) = \mathbb{1}$.