# Theory of Quantum Information Assignment 1

**Name:** Nate Stemen (20906566)
**Email:** `nate@stemen.email`

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All references to theorem/propositions/etc. are references to Watrous's book.

## Problem 1

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Phi \in \mathrm{CP}(\mathcal{X}, \mathcal{Y})$ be a completely positive map. Prove that there exists an operator $B \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y})$, for some choice of a complex Euclidean space $\mathcal{Z}$, such that

$$\Phi(X) = B(X \otimes \mathbb{1}_{\mathcal{Z}})B^*$$

for all $X \in \mathrm{L}(\mathcal{X})$. Identify a condition on the operator $B$ that is equivalent to $\Phi$ preserving trace.

**Solution.** By Proposition 2.18 we know that $\Phi^*$ is also a completely positive map $\Phi^* \in \mathrm{CP}(\mathcal{Y}, \mathcal{X})$. Now, using the Stinespring representation of $\Phi^*$ and Theorem 2.22, there exists an $A \in \mathrm{L}(\mathcal{Y}, \mathcal{X} \otimes \mathcal{Z})$ such that

$$\Phi^*(Y) = \mathrm{tr}_{\mathcal{Z}}(AYA^*). \tag{1}$$

Using the fact that $(\Phi^*)^* = \Phi$, and Equation 2.72 which gives us the dual representation of a Strinespring map we have

$$\Phi(X) = \Phi^{**}(X) = A^*(X \otimes \mathbb{1}_{\mathcal{Z}})A.$$

Finally, taking $B = A^*$ we see first $B \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y})$ and indeed

$$\Phi(X) = B(X \otimes \mathbb{1}_{\mathcal{Z}})B^*.$$

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To derive a condition on $B$ such that $\Phi$ preserves trace we set $\mathrm{tr}(X) = \mathrm{tr}(\Phi(X))$.

$$\begin{aligned} \mathrm{tr}(X) = \mathrm{tr}(\Phi(X)) &= \mathrm{tr}(B(X \otimes \mathbb{1}_{\mathcal{Z}})B^*) \\ &= \mathrm{tr}_{\mathcal{X}}(\mathrm{tr}_{\mathcal{Z}}(B^*B(X \otimes \mathbb{1}_{\mathcal{Z}}))) \\ &= \mathrm{tr}(\mathrm{tr}_{\mathcal{Z}}(B^*B)X) \end{aligned}$$

Thus, in order for this mapping to preserve trace we must have $\mathrm{tr}_{\mathcal{Z}}(B^*B) = \mathbb{1}_{\mathcal{X}}$.

Perhaps a simpler way of seeing this is using the fact that $\Phi$ preserving trace is equivalent to $\Phi^*$ being a unital map. Thus using eq. (1) we have

$$\Phi^*(\mathbb{1}_{\mathcal{Y}}) = \mathrm{tr}_{\mathcal{Z}}(B^*\mathbb{1}_{\mathcal{Y}}B) = \mathrm{tr}_{\mathcal{Z}}(B^*B) = \mathbb{1}_{\mathcal{X}}.$$

**Problem 2**

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be a positive (but not necessarily completely positive) map, and let $\Delta \in \mathrm{C}(\mathcal{Y})$ denote the completely dephasing channel with respect to the space $\mathcal{Y}$. Prove that $\Delta\Phi$ is completely positive.

**Solution.** Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Gamma$ with $E_{a,b}$ being the standard basis for $\mathrm{L}(\mathcal{X})$ and $\tilde{E}_{a,b}$ be the standard basis for $\mathrm{L}(\mathcal{Y})$. First, the Choi representation of $\Phi$ can be expanded:

$$\begin{aligned} J(\Phi) &= \sum_{a,b\in\Sigma} \Phi(E_{a,b}) \otimes E_{a,b} \\ &= \sum_{a,b\in\Sigma} \left[ \sum_{c,d\in\Gamma} \alpha_{c,d} \tilde{E}_{c,d} \right] \otimes E_{a,b} \\ &= \sum_{\substack{a,b\in\Sigma \\ c,d\in\Gamma}} \alpha_{c,d} \tilde{E}_{c,d} \otimes E_{a,b} \end{aligned}$$

We can now use the fact that $J(\Delta\Phi) = [\Delta \otimes \mathbb{1}_{\mathcal{X}}]J(\Phi)$ to write the Choi representation of $J(\Delta\Phi)$.

$$\begin{aligned} J(\Delta\Phi) &= [\Delta \otimes \mathbb{1}_{\mathcal{X}}]J(\Phi) \\ &= \sum_{\substack{a,b\in\Sigma \\ c,d\in\Gamma}} \alpha_{c,d} \Delta(\tilde{E}_{c,d}) \otimes E_{a,b} \\ &= \sum_{\substack{a,b\in\Sigma \\ c\in\Gamma}} \alpha_{c,c} \tilde{E}_{c,c} \otimes E_{a,b} \end{aligned}$$

With this we now aim to use Theorem 2.22 to prove $\Delta\Phi$ is completely positive by showing $J(\Delta\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$. First we derive a condition on the $\alpha_{c,d}$ coefficients using the fact that $\Phi$ is a positive map. Let $\{|c\rangle\}_{c\in\Gamma}$ be a basis for $\mathcal{Y}$, and since $E_{a,b}$ is a positive operator, so should $\Phi(E_{a,b})$.

$$\langle\tilde{c}|\Phi(E_{a,b})|\tilde{c}\rangle = \langle\tilde{c}| \sum_{c,d\in\Gamma} \alpha_{c,d} \tilde{E}_{c,d} |\tilde{c}\rangle = \sum_{c,d\in\Gamma} \alpha_{c,d} \langle\tilde{c}|c\rangle \langle d|\tilde{c}\rangle = \sum_{c,d\in\Gamma} \alpha_{c,d} \delta_{\tilde{c},c} \delta_{d,\tilde{c}} = \alpha_{\tilde{c},\tilde{c}}$$

Thus for all $c \in \Gamma$ we have $\alpha_{c,c} \geq 0$.

Now take $\{|c\rangle \otimes |a\rangle\}_{\substack{a\in\Sigma \\ c\in\Gamma}}$ to be a basis for $\mathcal{Y} \otimes \mathcal{X}$. We can now check if $J(\Delta\Phi) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ by showing $\langle c| \otimes \langle a| \, J(\Delta\Phi) \, |c\rangle \otimes |a\rangle \geq 0$.

$$\begin{aligned} \langle\tilde{c}| \otimes \langle\tilde{a}| \, J(\Delta\Phi) \, |\tilde{c}\rangle \otimes |\tilde{a}\rangle &= \sum_{\substack{a,b\in\Sigma \\ c\in\Gamma}} \alpha_{c,c} \langle\tilde{c}| \otimes \langle\tilde{a}| \; \tilde{E}_{c,c} \otimes E_{a,b} \; |\tilde{c}\rangle \otimes |\tilde{a}\rangle \\ &= \sum_{\substack{a,b\in\Sigma \\ c\in\Gamma}} \alpha_{c,c} \langle\tilde{c}|c\rangle \langle c|\tilde{c}\rangle \langle\tilde{a}|a\rangle \langle b|\tilde{a}\rangle \\ &= \alpha_{\tilde{c},\tilde{c}} \end{aligned}$$

Since we've already shown $\alpha_{c,c} \geq 0$ for all $c \in \Gamma$, this shows that $J(\Delta\Phi)$ is positive semidefinite by extending this argument to linear combinations of the basis elements.

## Problem 3

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Sigma$ be an alphabet. Suppose further that $\eta : \Sigma \to \mathrm{Pos}(\mathcal{X})$ is a function satisfying

$$\sum_{a\in\Sigma} \eta(a) \in \mathrm{D}(\mathcal{X}),$$

which is to say that $\eta$ represents an *ensemble of states*, and $u \in \mathcal{X} \otimes \mathcal{Y}$ is a purification of the average state of this ensemble:

$$\mathrm{tr}_{\mathcal{Y}}(uu^*) = \sum_{a\in\Sigma} \eta(a).$$

Prove that there exists a measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ for which

$$\eta(a) = \mathrm{tr}_{\mathcal{Y}} \left[ (\mathbb{1}_{\mathcal{X}} \otimes \mu(a))uu^* \right]$$

for all $a \in \Sigma$.

Solution completed in collaboration with Alev Orfi,[1] and Muhammad Usman Farooq.[2]

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**Solution.** The fact that $\mathrm{vec} : \mathrm{L}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y} \otimes \mathcal{X}$ is a bijection allows us to find an operator $A \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ such that $\mathrm{vec}(A) = u$. We thus have

$$\mathrm{tr}_{\mathcal{Y}}(uu^*) = \sum_{a\in\Sigma} \eta(a) = AA^*$$

using $\mathrm{tr}_{\mathcal{Y}}(\mathrm{vec}(C)\,\mathrm{vec}(B)^*) = CB^*$ for $C, B \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$. Combining Exercise 3 which says $\mathrm{im}(P) \subseteq \mathrm{im}(P + Q)$ for $P, Q \in \mathrm{Pos}(\mathcal{X})$ and the fact that $\eta(A) \in \mathrm{Pos}(\mathcal{X})$, we can write $\mathrm{im}(\eta(a)) \subseteq \mathrm{im}(\sum \eta(a)) = \mathrm{im}(AA^*) \subseteq \mathrm{im}(A)$. Now, we need a lemma—Lemma 2.30 from Watrous.

**Lemma.** For $D \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ we have the following equality of sets.

$$\{P \in \mathrm{Pos}(\mathcal{X}) : \mathrm{im}(P) \subseteq \mathrm{im}(D)\} = \{DQD^* : Q \in \mathrm{Pos}(\mathcal{X})\}$$

Thus since $\eta(a) \in \mathrm{Pos}(\mathcal{X})$ and $\mathrm{im}(\eta(a)) \subseteq \mathrm{im}(A)$, we can find a $Q_a \in \mathrm{Pos}(\mathcal{X})$ such that $\eta(a) = AQ_aA^*$. In particular since the transpose is a positive map $Q_a^{\mathsf{T}}$ is also positive and we take as definition $\mu(a) := Q_a^{\mathsf{T}}$.

We now show $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ is indeed a measurement. We've already shown $\mu(a)$ to be a positive semi-definite operator on $\mathcal{X}$, so we need to show $\mu$ resolves the identity.

$$\sum_{a\in\Sigma} \eta(a) = \sum_{a\in\Sigma} A\mu(a)^{\mathsf{T}}A^* = A\left[\sum_{a\in\Sigma} \mu(a)^{\mathsf{T}}\right]A^* = AA^*$$

This last equality follows from the definition of $A$, and in order for it to hold we must have $\sum_{a\in\Sigma} \mu(a)^{\mathsf{T}} = \mathbb{1}_{\mathcal{X}}$. Taking the transpose of both sides we have the required summation.

[1] akborfi@uwaterloo.ca
[2] mu7faroo@uwaterloo.ca

Lastly we must show these $\mu(a)$ relate to $\eta(a)$ in the proper way.

$$\begin{aligned}
\eta(a) &= A\mu(a)^{\mathsf{T}} A^* \\
&= \operatorname{tr}_{\mathcal{Y}} \left[\operatorname{vec}(A\mu(a)^{\mathsf{T}}) \operatorname{vec}(A)^*\right] \\
&= \operatorname{tr}_{\mathcal{Y}} \left[\operatorname{vec}(\mathbb{1}_{\mathcal{X}} A\mu(a)^{\mathsf{T}}) \operatorname{vec}(A)^*\right] \\
&= \operatorname{tr}_{\mathcal{Y}} \left[(\mathbb{1}_X \otimes \mu(a)) \operatorname{vec}(A) \operatorname{vec}(A)^*\right] \\
&= \operatorname{tr}_{\mathcal{Y}} \left[(\mathbb{1}_X \otimes \mu(a)) uu^*\right]
\end{aligned}$$

**Problem 4**

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces and let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel. Prove that the following two statements are equivalent:

(a) There exists a complex Euclidean space $\mathcal{W}$, a state $\sigma \in \mathrm{D}(\mathcal{Y} \otimes \mathcal{W})$, and a channel $\Psi \in \mathrm{C}(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$ so that

$$\Phi(X) = \left[\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi\right](\sigma \otimes X)$$

for all $X \in \mathrm{L}(\mathcal{X})$.

(b) There exists a density operator $\rho \in \mathrm{D}(\mathcal{Y})$ for which

$$\operatorname{tr}_{\mathcal{Z}}(J(\Phi)) = \rho \otimes \mathbb{1}_{\mathcal{X}}.$$

Solution completed in collaboration with Mohammad Ayyash,[3] and Nicholas Zutt.[4]

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**Solution.** (a) $\implies$ (b) To start let us take $\mathcal{X} = \mathbb{C}^{\Sigma}, \mathcal{Y} = \mathbb{C}^{\Gamma}$ and $\mathcal{W} = \mathbb{C}^{\Pi}$. We can now expand $\sigma$ in the tensor product basis of $\mathrm{D}(\mathcal{Y} \otimes \mathcal{W})$ as

$$\sigma = \sum_{\substack{c,d \in \Gamma \\ e,f \in \Pi}} \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes E^{\mathcal{W}}_{e,f}. \tag{2}$$

Where we use $E^{\mathcal{V}}_{a,b}$ to be the standard matrix on space $\mathcal{V}$. This allows us to expand $\Phi$ as follows.

$$\Phi(X) = \sum_{\substack{c,d \in \Gamma \\ e,f \in \Pi}} \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \Psi(E^{\mathcal{W}}_{e,f} \otimes X)$$

Now we compute the Choi representation of $\Phi$, and then trace out $\mathcal{Z}$. If there are no limits on the summation, assume it is the same limits as the previous summation *with* limits.

$$J(\Phi) = \sum_{\substack{a,b \in \Sigma \\ c,d \in \Gamma \\ e,f \in \Pi}} \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \Psi(E^{\mathcal{W}}_{e,f} \otimes E^{\mathcal{X}}_{a,b}) \otimes E^{\mathcal{X}}_{a,b}$$

$$\begin{aligned}
\operatorname{tr}_{\mathcal{Z}}(J(\Phi)) &= \sum \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \operatorname{tr}\left[\Psi(E^{\mathcal{W}}_{e,f} \otimes E^{\mathcal{X}}_{a,b})\right] \otimes E^{\mathcal{X}}_{a,b} \\
&= \sum \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \operatorname{tr}\left[E^{\mathcal{W}}_{e,f} \otimes E^{\mathcal{X}}_{a,b}\right] \otimes E^{\mathcal{X}}_{a,b} && (\Psi \text{ preserves trace}) \\
&= \sum \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \operatorname{tr}\left[E^{\mathcal{W}}_{e,f}\right] \operatorname{tr}\left[E^{\mathcal{X}}_{a,b}\right] \otimes E^{\mathcal{X}}_{a,b} && (\operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B) \\
&= \sum \sigma_{c,d,e,f}\ \delta_{e,f}\delta_{a,b}\ E^{\mathcal{Y}}_{c,d} \otimes E^{\mathcal{X}}_{a,b} && (\operatorname{tr} E^{\mathcal{V}}_{a,b} = \delta_{a,b}) \\
&= \sum_{\substack{a \in \Sigma \\ c,d \in \Gamma \\ e \in \Pi}} \sigma_{c,d,e,e} E^{\mathcal{Y}}_{c,d} \otimes E^{\mathcal{X}}_{a,a} \\
&= \sum_{\substack{c,d \in \Gamma \\ e \in \Pi}} \sigma_{c,d,e,e} E^{\mathcal{Y}}_{c,d} \otimes \left[\sum_{a \in \Sigma} E^{\mathcal{X}}_{a,a}\right] = \operatorname{tr}_{\mathcal{W}}(\sigma) \otimes \mathbb{1}_{\mathcal{X}}
\end{aligned}$$

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[3]mmayyash@uwaterloo.ca

[4]nzutt@uwaterloo.ca

The last step follows from taking the trace of eq. (2):

$$\mathrm{tr}_{\mathcal{W}}(\sigma) = \sum_{\substack{c,d\in\Gamma \\ e,f\in\Pi}} \sigma_{c,d,e,f} E^{\mathcal{Y}}_{c,d} \otimes \mathrm{tr}\left[E^{\mathcal{W}}_{e,f}\right] = \sum_{\substack{c,d\in\Gamma \\ e\in\Pi}} \sigma_{c,d,e,e} E^{\mathcal{Y}}_{c,d}$$

Thus taking $\rho := \mathrm{tr}_{\mathcal{W}}(\sigma)$ we have $\mathrm{tr}_{\mathcal{Z}}(J(\Phi)) = \rho \otimes \mathbb{1}_{\mathcal{X}}$. The fact that $\rho \in \mathrm{D}(\mathcal{Y})$ follows from the fact that the trace is completely positive, and clearly $\mathrm{tr}(\rho) = \mathrm{tr}(\mathrm{tr}_{\mathcal{W}}(\sigma)) = \mathrm{tr}_{\mathcal{Y}\otimes\mathcal{W}}(\sigma) = 1$.
(b) $\Longrightarrow$ (a) Let $u \in \mathcal{Y} \otimes \mathcal{W} \otimes \mathcal{X}$ be a purification of $\rho \otimes \mathbb{1}_{\mathcal{X}}$. That is $\mathrm{tr}_{\mathcal{W}}(uu^*) = \rho \otimes \mathbb{1}_X = \mathrm{tr}_{\mathcal{Z}}(J(\Phi))$. Further we have $\mathrm{tr}_{\mathcal{W}\otimes\mathcal{X}}(uu^*) = \dim(\mathcal{X}) \cdot \rho = \mathrm{tr}_{\mathcal{X}}\,\mathrm{tr}_{\mathcal{Z}}(J(\Phi)) = \mathrm{tr}_{\mathcal{Z}\otimes\mathcal{X}}(J(\Phi))$. Proposition 2.29 with $P = J(\Phi)$ gives us the existence of a channel $\Lambda \in \mathrm{C}(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z} \otimes \mathcal{X})$ such that

$$\left(\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Lambda\right)(uu^*) = J(\Phi).$$

We can now use

$$\Phi(X) = \mathrm{tr}_{\mathcal{X}}\left(J(\Phi)(\mathbb{1}_{\mathcal{Y}\otimes\mathcal{Z}} \otimes X^{\mathsf{T}})\right)$$

to recover the action of $\Phi$ from the Choi representation.

$$\Phi(X) = \mathrm{tr}_{\mathcal{X}}\left[\left([\mathbb{1}_{\mathcal{Y}} \otimes \Lambda](uu^*)\right)\left(\mathbb{1}_{\mathcal{Y}\otimes\mathcal{Z}} \otimes X^{\mathsf{T}}\right)\right]$$

We can now trace out $\mathcal{X}$ after applying $\Lambda$ to give a channel $\Psi := \mathrm{tr}_{\mathcal{X}} \circ \Lambda$ which is an element of $\mathrm{C}(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$. This allows us to write the final form of $\Phi$ as

$$\Phi(X) = \left(\mathbb{1}_{\mathcal{Y}} \otimes \Psi\right)\left(\underbrace{\mathrm{tr}_{\mathcal{X}}(uu^*)}_{\sigma} \otimes X^{\mathsf{T}}\right).$$

I have no idea how to get rid of the tranpose.