Theory of Quantum Information Assignment 2

Name: Nate Stemen (20906566) **Due:** Nov 5, 2021
Email: nate@stemen.email **Course:** QIC 820

Problem 1

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are complex Euclidean spaces and $M \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ is a given operator. Define a map $\Phi \in \mathrm{T}(\mathcal{X} \oplus \mathcal{Y})$ as

$$\Phi\begin{pmatrix} X & Z \\ W & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

for all $X \in \mathrm{L}(\mathcal{X})$, $Y \in \mathrm{L}(\mathcal{Y})$, $Z \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$, and $W \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$ (i.e., Φ zeroes out the off-diagonal blocks of a 2×2 block operator of the form suggested in the equation), and consider the semidefinite program

$$\left(\Phi, \frac{1}{2}\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & 0 \\ 0 & \mathbb{1}_{\mathcal{Y}} \end{pmatrix}\right).$$

(a) Express the primal and dual problems associated with this semidefinite program in simple, human-readable terms. (There is no single, well-defined answer to this part of the problem—just do your best to make the primal and dual problems look as simple and elegant as possible.)
(b) Prove that strong duality holds for this semidefinite program.
(c) What is the optimal value of this semidefinite program?

Solution completed in collaboration with Alev Orfi,[1] and Muhammad Usman Farooq.[2]

Solution. (a) Given the nice form of Φ we can start by doing some algebra to simplify the program. Starting with the definition of a semidefinite program we have

$$\begin{aligned} \underset{X}{\text{maximize:}} \quad & \left\langle \tfrac{1}{2}\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, X \right\rangle \\ \text{subject to:} \quad & \Phi(X) = \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & 0 \\ 0 & \mathbb{1}_{\mathcal{Y}} \end{pmatrix} \\ & X \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y}). \end{aligned}$$

Taking $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ together with the condition that $\Phi(X) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & 0 \\ 0 & \mathbb{1}_{\mathcal{Y}} \end{pmatrix}$ we see that $A = \mathbb{1}_{\mathcal{X}}$ and $B = \mathbb{1}_{\mathcal{Y}}$. Since $X \in \mathrm{Pos}(\mathcal{X})$ it must also be hermitian ($X = X^*$), and hence $C = B^*$ making $X = \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & B \\ B^* & \mathbb{1}_{\mathcal{Y}} \end{pmatrix}$. We can now expand out the inner product we are

[1] akborfi@uwaterloo.ca
[2] mu7faroo@uwaterloo.ca

trying to maximize as follows (temporarily omitting the $\frac{1}{2}$):

$$\begin{aligned}\left\langle \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & B \\ B^* & \mathbb{1}_{\mathcal{Y}} \end{pmatrix} \right\rangle &= \operatorname{tr}\left[\begin{pmatrix} MB^* & M \\ M^* & M^*B \end{pmatrix}\right] \\ &= \operatorname{tr}(MB^*) + \operatorname{tr}(M^*B) \\ &= \langle B, M\rangle + \langle B^*, M^*\rangle\end{aligned}$$

This allows us to rewrite the primal problem as

$$\begin{aligned}\underset{B}{\text{maximize:}} \quad & \tfrac{1}{2}\langle B, M\rangle + \tfrac{1}{2}\langle B^*, M^*\rangle \\ \text{subject to:} \quad & \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & B \\ B^* & \mathbb{1}_{Y} \end{pmatrix} \in \operatorname{Pos}(\mathcal{X} \oplus \mathcal{Y}) \\ & B \in \mathrm{L}(\mathcal{Y}, \mathcal{X}).\end{aligned}$$

In order to understand the dual problem we have to first know what Φ^* is. This can be calculated by the definition of a adjoint map: $\langle \Phi(X), Y\rangle = \langle X, \Phi^*(Y)\rangle$.

$$\left\langle \Phi\begin{pmatrix} X & Z \\ W & Y \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\rangle = \operatorname{tr}\left(\begin{pmatrix} X^*A & \cdot \\ \cdot & Y^*D \end{pmatrix}\right) = \operatorname{tr}(X^*A) + \operatorname{tr}(Y^*D) = \langle X, A\rangle + \langle Y, D\rangle$$
$$\left\langle \begin{pmatrix} X & Z \\ W & Y \end{pmatrix}, \Phi^*\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\rangle = \operatorname{tr}\left(\begin{pmatrix} X^*A' + W^*C' & \cdot \\ \cdot & Z^*B' + Y^*D' \end{pmatrix}\right)$$

In order for this to be equal to $\operatorname{tr}(X^*A) + \operatorname{tr}(Y^*D)$ we must have $A' = A$, $D' = D$, as well as $C' = 0 = B'$. This gives us the action of Φ^* as $\Phi^*\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and hence $\Phi = \Phi^*$.

To simplify the dual problem we first write it in it's full generality:

$$\begin{aligned}\underset{Y}{\text{minimize:}} \quad & \left\langle \begin{pmatrix} \mathbb{1}_{\mathcal{X}} & 0 \\ 0 & \mathbb{1}_{\mathcal{Y}} \end{pmatrix}, Y \right\rangle \\ \text{subject to:} \quad & \Phi(Y) \geq \tfrac{1}{2}\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} \\ & Y \in \operatorname{Herm}(\mathcal{X} \oplus \mathcal{Y}).\end{aligned}$$

With a little bit of algebra, together with the fact that $Y \in \operatorname{Herm}(\mathcal{X} \oplus \mathcal{Y})$ this problem transforms to

$$\begin{aligned}\underset{A,D}{\text{minimize:}} \quad & \operatorname{tr}(A) + \operatorname{tr}(D) \\ \text{subject to:} \quad & \begin{pmatrix} A & -\frac{1}{2}M \\ -\frac{1}{2}M^* & D \end{pmatrix} \in \operatorname{Pos}(\mathcal{X} \oplus \mathcal{Y}) \\ & A \in \operatorname{Herm}(\mathcal{X}) \\ & D \in \operatorname{Herm}(\mathcal{Y}).\end{aligned}$$

First we redefine $\tilde{A} = \frac{1}{2}A$ and $\tilde{D} = \frac{1}{2}D$. We can also show $\tilde{A} \in \operatorname{Pos}(\mathcal{X})$ as follows. Since $T = \begin{pmatrix} \tilde{A} & -M \\ -M^* & \tilde{D} \end{pmatrix} \in \operatorname{Pos}(\mathcal{X} \oplus \mathcal{Y})$ we know $x^*Tx \geq 0$ for all $x \in \mathcal{X} \oplus \mathcal{Y}$. In particular this is true for $x = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$ separately. This shows $x_0^*\tilde{A}x_0 \geq 0$ for all $x_0 \in \mathcal{X}$ and $y_0^*\tilde{D}y_0 \geq 0$ for all $y_0 \in \mathcal{Y}$. Thus our final dual problem has simplified to

$$\begin{aligned}\underset{A,D}{\text{minimize:}} \quad & \tfrac{1}{2}\operatorname{tr}(A) + \tfrac{1}{2}\operatorname{tr}(D) \\ \text{subject to:} \quad & \begin{pmatrix} A & -M \\ -M^* & D \end{pmatrix} \in \operatorname{Pos}(\mathcal{X} \oplus \mathcal{Y}) \\ & A \in \operatorname{Pos}(\mathcal{X}) \\ & D \in \operatorname{Pos}(\mathcal{Y}).\end{aligned}$$

(b) We can now apply Lemma 3.18 to our primal problem. This means $\begin{pmatrix} \mathbb{1}_{\mathcal{X}} & B \\ B^* & \mathbb{1}_{\mathcal{Y}} \end{pmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y})$ if and only if $B = \sqrt{\mathbb{1}_{\mathcal{X}}} K \sqrt{\mathbb{1}_{\mathcal{Y}}} = K$ for some $K \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ with $\|K\| \leq 1$. Thus B is bounded $\|B\| \leq 1$ and since M is fixed α (optimum value for primal problem) must be finite. In order to use Slater's theorem for semidefinite programs I need to show there exists a $Y \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{Y})$ such that $\Phi^*(Y) \geq \frac{1}{2}\left(\begin{smallmatrix} 0 & M \\ M^* & 0 \end{smallmatrix}\right)$. That is we need to show for $A \in \mathrm{Herm}(\mathcal{X})$ and $D \in \mathrm{Herm}(\mathcal{Y})$ that

$$\Phi\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) - \frac{1}{2}\left(\begin{smallmatrix} 0 & M \\ M^* & 0 \end{smallmatrix}\right) \approx \begin{pmatrix} A & -M \\ -M^* & D \end{pmatrix} \in \mathrm{Pos}(\mathcal{X} \oplus \mathcal{Y}).$$

I've used $\approx$ to get rid of the halves. This if the exact condition we have in the dual problem, but I'm stuck as to where to go from here.

(c) The optimal value for this semidefinite program is the spectral norm: $\|M\|_1$.

Problem 2

Let $\mathcal{X}$ be a complex Euclidean space, and define

$$\delta(P, Q) = \sqrt{\operatorname{tr}(P) + \operatorname{tr}(Q) - 2\operatorname{F}(P, Q)}$$

for all positive semidefinite operators $P, Q \in \operatorname{Pos}(\mathcal{X})$. Prove that δ satisfies these three properties:

(a) $\delta(P, Q) \geq 0$ for all $P, Q \in \operatorname{Pos}(\mathcal{X})$, with $\delta(P, Q) = 0$ if and only if $P = Q$.

(b) $\delta(P, Q) = \delta(Q, P)$ for all $P, Q \in \operatorname{Pos}(\mathcal{X})$.

(c) $\delta(P, Q) \leq \delta(P, R) + \delta(R, Q)$ for all $P, Q, R \in \operatorname{Pos}(\mathcal{X})$.

(These are the three defining properties of a *metric*.)

Hint: to prove that property (c) holds, first prove that if $\mathcal{Y}$ is a complex Euclidean space with $\dim(\mathcal{Y}) \geq \dim(\mathcal{X})$, and $u \in \mathcal{X} \otimes \mathcal{Y}$ is any vector satisfying $\operatorname{tr}_{\mathcal{Y}}(uu^*) = P$, then

$$\delta(P, Q) = \min_{v \in \mathcal{X} \otimes \mathcal{Y}} \left\{ \|u - v\| : \operatorname{tr}_{\mathcal{Y}}(vv^*) = Q \right\}.$$

Solution. (a) By the symmetry of δ (proved next) we can without loss of generality take $\operatorname{tr}(P) \geq \operatorname{tr}(Q)$. Then, because $\operatorname{tr}(X) \geq 0$ for all $X \in \operatorname{Pos}(\mathcal{X})$ we have

$$\begin{aligned}
\left(\sqrt{\operatorname{tr}(P)} - \sqrt{\operatorname{tr}(Q)}\right)^2 &\geq 0 \\
\operatorname{tr}(P) + \operatorname{tr}(Q) - 2\sqrt{\operatorname{tr}(P)\operatorname{tr}(Q)} &\geq 0 \\
\operatorname{tr}(P) + \operatorname{tr}(Q) &\geq 2\sqrt{\operatorname{tr}(P)\operatorname{tr}(Q)} \\
&\geq 2\operatorname{F}(P, Q) && \text{(Proposition 3.12:6)}
\end{aligned}$$

By the last inequality we have that the radicand is greater than or equal to 0, and hence $\delta(P, Q) \geq 0$. In order for $\delta(P, Q) = 0$, we must have both $\operatorname{tr}(P) = \operatorname{tr}(Q)$ and $Q = \lambda P$ again by proposition 3.12:6. Then we have $\operatorname{tr}(P) = \operatorname{tr}(\lambda P) = \lambda \operatorname{tr}(P)$ and hence $\lambda = 1$ implying $P = Q$.

(b) The symmetry of δ follows immediately from Proposition 3.12:2 which states that $\operatorname{F}(P, Q) = \operatorname{F}(Q, P)$.

$$\delta(P, Q) = \sqrt{\operatorname{tr}(P) + \operatorname{tr}(Q) - 2\operatorname{F}(P, Q)} = \sqrt{\operatorname{tr}(Q) + \operatorname{tr}(P) - 2\operatorname{F}(Q, P)} = \delta(Q, P)$$

(c) We start by proving the hint. Let $u \in \mathcal{X} \otimes \mathcal{Y}$ be a purification of P.

$$\begin{aligned}
\delta(P, Q)^2 &= \min_v \left\{ \langle u - v, u - v \rangle : \operatorname{tr}_{\mathcal{Y}}(vv^*) = Q \right\} \\
&= \min_v \left\{ \|u\|^2 + \|v\|^2 - 2\operatorname{Re}(\langle u, v \rangle) \right\} \\
&= \operatorname{tr}(P) + \operatorname{tr}(Q) - 2\max_v \left\{ \operatorname{Re}(\langle u, v \rangle) \right\} \\
&\leq \operatorname{tr}(P) + \operatorname{tr}(Q) - 2\max_v \left\{ |\langle u, v \rangle| \right\} \\
&= \operatorname{tr}(P) + \operatorname{tr}(Q) - 2\operatorname{F}(P, Q) && \text{(Uhlmann's theorem)}
\end{aligned}$$

Thus in order for the hint to be true (which it has to, that would be illegal otherwise) we must have $\max_v \{\mathrm{Re}(\langle u, v\rangle)\} = \max_v \{|\langle u, v\rangle|\}$ which I cannot see why must be true.

With the hint proved, let $p, q, r \in \mathcal{X} \otimes \mathcal{Y}$ be a purifications of P, Q and R respectively.

$$\begin{aligned}
\delta(P, Q) &= \min_q \{\|p - q\|\} \\
&= \min_q \{\|p - q - r + r\|\} \\
&\leq \min_q \{\|p - r\| + \|r - q\|\} \\
&\leq \min_r \{\|p - r\|\} + \min_q \{\|r - q\|\} \\
&= \delta(P, R) + \delta(R, Q)
\end{aligned}$$

Problem 3

Let $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ be a map, for complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$. Prove that

$$|||\Phi|||_1 = \max_{\rho_0,\rho_1 \in \mathrm{D}(\mathcal{X})} \left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}\right) J(\Phi) \left(\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}\right) \right\|_1.$$

Solution completed in collaboration with Mohammad Ayyash,[3] and Nicholas Zutt.[4]

Solution. We begin with a lemma.

Lemma. Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Pi$ be complex euclidean spaces. For all $A, B \in \mathrm{L}(\mathcal{X})$ and $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ we have

$$(\mathbb{1}_{\mathcal{Y}} \otimes A^\intercal) J(\Phi) (\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}) = \left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}\right)(\mathrm{vec}(A)\,\mathrm{vec}(B)^*).$$

Proof. We start by expanding the Choi matrix using the definition: $J(\Phi) = \sum_{a,b\in\Sigma} \Phi(E_{a,b}) \otimes E_{a,b}$.

$$(\mathbb{1}_{\mathcal{Y}} \otimes A^\intercal) J(\Phi) (\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}) = \sum_{a,b\in\Sigma} \Phi(E_{a,b}) \otimes A^\intercal E_{a,b} \overline{B}$$

Let's calculate $A^\intercal E_{a,b}\overline{B}$ expanding A and B in the standard basis as $A = \sum_{a,b\in\Sigma} \alpha_{a,b} E_{a,b}$ and $B = \sum_{a,b\in\Sigma} \beta_{a,b} E_{a,b}$.

$$\begin{aligned}
A^\intercal E_{a,b}\overline{B} &= \left[\sum_{c,d\in\Sigma} \alpha_{d,c} E_{c,d}\right] E_{a,b} \left[\sum_{e,f\in\Sigma} \overline{\beta_{e,f}} E_{e,f}\right] \\
&= \sum_{c,d,e,f\in\Sigma} \alpha_{d,c}\overline{\beta_{e,f}}\, E_{c,d}\, E_{a,b}\, E_{e,f} \\
&= \sum_{c,f\in\Sigma} \alpha_{a,c}\overline{\beta_{b,f}} E_{c,f}
\end{aligned}$$

Thus together we have

$$\begin{aligned}
(\mathbb{1}_{\mathcal{Y}} \otimes A^\intercal) J(\Phi) (\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}) &= \sum_{a,b,c,f\in\Sigma} \alpha_{a,c}\, \overline{\beta_{b,f}}\, \Phi(E_{a,b}) \otimes E_{c,f} \\
&= \left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}\right) \left[\sum_{a,b,c,f\in\Sigma} \alpha_{a,c}\, \overline{\beta_{b,f}}\, E_{a,b} \otimes E_{c,f}\right]
\end{aligned}$$

We we must show the argument to $\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}$ is equal to $\mathrm{vec}(A)\,\mathrm{vec}(B)^*$.

$$\begin{aligned}
\sum_{a,b,c,f\in\Sigma} \alpha_{a,c}\, \overline{\beta_{b,f}}\, E_{a,b} \otimes E_{c,f} &= \left[\sum_{a,c\in\Sigma} \alpha_{a,c}\, e_a \otimes e_c\right] \left[\sum_{b,f\in\Sigma} \overline{\beta_{b,f}}\, e_b^* \otimes e_f^*\right] \\
&= \left[\sum_{a,c\in\Sigma} \alpha_{a,c}\, e_a \otimes e_c\right] \left[\sum_{b,f\in\Sigma} \beta_{b,f}\, e_b \otimes e_f\right]^* \\
&= \mathrm{vec}(A)\,\mathrm{vec}(B)^*
\end{aligned}$$

As desired.

---

[3] mmayyash@uwaterloo.ca
[4] nzutt@uwaterloo.ca

By Proposition 3.44:1 we know the completely bounded trace norm can be written $|||\Phi|||_1 = \max\left\{\left\|(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uv^*)\right\|_1 : u, v \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X})\right\}$. Applying the above lemma to this statement transforms it to

$$|||\Phi|||_1 = \max_{A,B \in \mathrm{L}(\mathcal{X})} \left\{ \left\| (\mathbb{1}_{\mathcal{Y}} \otimes A^{\mathsf{T}}) J(\Phi) \big(\mathbb{1}_{\mathcal{Y}} \otimes \overline{B}\big) \right\|_1 : \|A\|_2 = 1 = \|B\|_2 \right\}$$

because $\mathrm{vec}(A) \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X})$ implies $\sqrt{\sum_{a,b \in \Sigma} \alpha_{a,b}^2} = 1$ which is exactly the expression for $\|A\|_2$. Applying the polar decomposition to A (and B) we have $A = UP$ where $U \in \mathrm{U}(\mathcal{X})$ and $P \in \mathrm{Pos}(\mathcal{X})$. Using the fact that $\|X\|_2 = \sqrt{\mathrm{tr}(X^*X)}$ we have

$$1 = \|A\|_2 = \mathrm{tr}(A^*A) = \mathrm{tr}(P^*U^*UP) = \mathrm{tr}(P^*P).$$

Since the unitary component does not affect the norm $\|\cdot\|_2$, we are ranging over positive semi-definite operators $\rho := P^*P$ such that $\mathrm{tr}\,\rho = 1$. That is, the maximum is attained when ranging over density operators. We can then express A in terms of this density operator as $A = \sqrt{\rho}$.

Using all these facts, together with the fact that $\sqrt{\rho}^{\mathsf{T}}$ can always be written as $\sqrt{\rho'}$ we have the desired equality:

$$|||\Phi|||_1 = \max_{\rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})} \left\| (\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_0}) J(\Phi) (\mathbb{1}_{\mathcal{Y}} \otimes \sqrt{\rho_1}) \right\|.$$