Theory of Quantum Information Assignment 4

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Problem 1

Let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ be a channel, for complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$. Prove that the following three statements are equivalent:

(a) For every complex Euclidean space $\mathcal{Z}$ and every state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Z})$, we have

$$(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho) \in \mathrm{SepD}(\mathcal{Y} : \mathcal{Z}).$$

(b) $J(\Phi) \in \mathrm{Sep}(\mathcal{Y} : \mathcal{X})$.

(c) There exists an alphabet Σ, a measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$, and a collection of states $\{\sigma_a : a \in \Sigma\} \subseteq \mathrm{D}(\mathcal{Y})$ such that

$$\Phi(X) = \sum_{a \in \Sigma} \langle \mu(a) | X \rangle \, \sigma_a$$

for all $X \in \mathrm{L}(\mathcal{X})$.

Solution. First we show (a) $\implies$ (b). Since (a) holds for all $\mathcal{Z}$ and $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Z})$, we can take $\mathcal{Z} = \mathcal{X}$, and $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{X})$ to be equal to $\mathrm{vec}(\mathbb{1}_{\mathcal{X}}) \mathrm{vec}(\mathbb{1}_{\mathcal{X}})^* / \dim(\mathcal{X})$. Thus we have

$$(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(\mathrm{vec}(\mathbb{1}_X) \, \mathrm{vec}(\mathbb{1}_{\mathcal{X}})^* / \dim(\mathcal{X})) \in \mathrm{SepD}(\mathcal{Y} : \mathcal{X})$$

and by the linearity of $\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})}$ we can pull $\frac{1}{\dim(\mathcal{X})}$ out. Using the cone relation that $\mathrm{cone}(\mathrm{SepD}(\mathcal{W})) = \mathrm{Sep}(\mathcal{W})$, we can move the (positive) constant to the other side to get

$$(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(\mathrm{vec}(\mathbb{1}_X) \, \mathrm{vec}(\mathbb{1}_{\mathcal{X}})^*) =: J(\Phi) \in \mathrm{Sep}(\mathcal{Y} : \mathcal{X}).$$

We now prove (b) $\implies$ (c). Since $J(\Phi)$ is separable, we can write it as

$$J(\Phi) = \sum_{a \in \Sigma} A_a \otimes B_a$$

with Σ being an alphabet, $A_a \in \mathrm{Pos}(\mathcal{Y})$, and $B_a \in \mathrm{Pos}(\mathcal{X})$ for all $a \in \Sigma$. Combining this fact together with Equation 2.66 from the text ($\Phi(X) = \mathrm{tr}_{\mathcal{X}}(J(\Phi) \, \mathbb{1}_{\mathcal{Y}} \otimes X^{\mathsf{T}})$) we have the following manipulations.

$$\begin{aligned}
\Phi(X) &= \mathrm{tr}_{\mathcal{X}}\left(\sum_{a \in \Sigma} A_a \otimes B_a [\mathbb{1}_{\mathcal{Y}} \otimes X^{\mathsf{T}}] \right) \\
&= \sum_{a \in \Sigma} A_a \, \mathrm{tr}(B_a X^{\mathsf{T}}) \\
&= \sum_{a \in \Sigma} \frac{A_a}{\mathrm{tr}(A_a)} \mathrm{tr}\big(\mathrm{tr}(A_a) \overline{B_a} X\big) && (\mathrm{tr}(B_A X^{\mathsf{T}}) = \mathrm{tr}((B_a X^{\mathsf{T}})^{\mathsf{T}}) = \mathrm{tr}(B_a^{\mathsf{T}} X) = \mathrm{tr}(\overline{B_a} X)) \\
&= \sum_{a \in \Sigma} \frac{A_a}{\mathrm{tr}(A_a)} \big\langle \mathrm{tr}(A_a) B_a^{\mathsf{T}} \big| X \big\rangle && (\mathrm{tr}(A_a) \in \mathbb{R}_{\geq 0})
\end{aligned}$$

Defining $\sigma_a := \frac{A_a}{\operatorname{tr}(A_a)} \in \mathrm{D}(\mathcal{Y})$ along with $\mu(a) := \operatorname{tr}(A_a)B_a^{\mathsf{T}} \in \operatorname{Pos}(\mathcal{X})$ allows us to express Φ as

$$\Phi(X) = \sum_{a\in\Sigma} \langle \mu(a)|X\rangle \, \sigma_a.$$

What's left to show is that $\mu : \Sigma \to \operatorname{Pos}(\mathcal{X})$ is indeed a measurement. First $\mu(a) = \operatorname{tr}(A_a)B_a^{\mathsf{T}} \in \operatorname{Pos}(\mathcal{X})$ because the transpose preserves positive semidefinite operators. To show the completeness relation holds we will use the fact that $\operatorname{tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$ for all channels Φ. Since the Choi representation is separable this gives us a condition on the A_a's and B_a's.

$$\operatorname{tr}_{\mathcal{Y}}(J(\Phi)) = \sum_{a\in\Sigma} \operatorname{tr}_{\mathcal{Y}}(A_a \otimes B_a) = \sum_{a\in\Sigma} \operatorname{tr}(A_a)B_a = \mathbb{1}_{\mathcal{X}}$$

Summing up the measurement operators we then have

$$\sum_{a\in\Sigma} \mu(a) = \sum_{a\in\Sigma} \operatorname{tr}(A_a)B_a^{\mathsf{T}} = \left(\sum_{a\in\Sigma} \operatorname{tr}(A_a)B_a\right)^{\mathsf{T}} = \mathbb{1}_{\mathcal{X}}^{\mathsf{T}} = \mathbb{1}_{\mathcal{X}}.$$

Thus μ is indeed a measurement.

Finally, we show (c) $\implies$ (a) to complete the equivalences.

$$\begin{aligned}(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho) &= \sum_{a\in\Sigma} \sigma_a \otimes \operatorname{tr}_{\mathcal{X}}\Big((\mu(a) \otimes \mathbb{1}_{\mathcal{Z}})(\rho)\Big) \\ &= \sum_{a\in\Sigma} \sigma_a \otimes \operatorname{tr}_{\mathcal{X}}\left[\left(\sqrt{\mu(a)} \otimes \mathbb{1}_{\mathcal{Z}}\right)\rho\left(\sqrt{\mu(a)} \otimes \mathbb{1}_{\mathcal{Z}}\right)\right]\end{aligned}$$

We'd like to show this is seperable, and clearly $\sigma_a \in \operatorname{Pos}(\mathcal{Y})$, so the first term is taken care of. The second term is Hermitian since the adjoint commutes with partial trace and the fact that ρ is Hermitian Second, the eigenvalues of the second term are all positive as the register Z is now in the state

$$\frac{\operatorname{tr}_{\mathcal{X}}[(\mu(a) \otimes \mathbb{1}_{\mathcal{Z}})\rho]}{\langle \mu(a)|\rho[\mathcal{X}]\rangle}.$$

Thus, because it is Hermitian with positive eigenvalues, it is positive semidefinite and hence $(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho) \in \operatorname{Sep}(\mathcal{Y} : \mathcal{Z})$. To see this is also a density operator, we can see that tracing over $\mathcal{Y}$ yields only the second tensor terms. We can then take the sum inside $\operatorname{tr}_{\mathcal{X}}$, use the completeness relation for μ to obtain $[\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Z}}](\rho) = \rho$. Finally $\operatorname{tr}(\rho) = 1$, so this is indeed a density operator, and indeed $(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho) \in \operatorname{SepD}(\mathcal{Y} : \mathcal{Z})$.

Problem 2

For any channel Ξ, define the *minimum output entropy* of Ξ as

$$\mathrm{H}_{\text{min}}(\Xi) = \min_{\rho} \mathrm{H}(\Xi(\rho)),$$

where the minimum is over all density operator inputs to Ξ.
Suppose that $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ and $\Psi \in \mathrm{C}(\mathcal{Z}, \mathcal{W})$ are channels, for complex Euclidean spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{W}$, and assume that $J(\Phi) \in \mathrm{Sep}(\mathcal{Y} : \mathcal{X})$. Prove that

$$\mathrm{H}_{\text{min}}(\Phi \otimes \Psi) = \mathrm{H}_{\text{min}}(\Phi) + \mathrm{H}_{\text{min}}(\Psi).$$

Remark: the equality is not true in general without the assumption $J(\Phi) \in \mathrm{Sep}(\mathcal{Y} : \mathcal{X})$; a correct answer must make use of this assumption.

Solution completed in collaboration with Alev Orfi,[1] and Muhammad Usman Farooq.[2]

Solution. We first show $\mathrm{H}_{\text{min}}(\Phi \otimes \Psi) \leq \mathrm{H}_{\text{min}}(\Phi) + \mathrm{H}_{\text{min}}(\Psi)$. To see this take $\rho = a \otimes b$, with $a \in \mathrm{D}(\mathcal{X})$ and $b \in \mathrm{D}(\mathcal{Y})$ and compute the entropy with respect to product states.

$$\mathrm{H}(\Phi \otimes \Psi(a \otimes b)) = \mathrm{H}(\Phi(a) \otimes \Psi(b)) = \mathrm{H}(\Phi(a)) + \mathrm{H}(\Psi(b))$$

We of course have

$$\begin{aligned}
\mathrm{H}_{\text{min}}(\Phi \otimes \Psi) &:= \min_{\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Z})} \mathrm{H}(\Phi \otimes \Psi(\rho)) \\
&\leq \min_{\substack{\rho = a \otimes b \\ a \in \mathrm{D}(\mathcal{X}) \\ b \in \mathrm{D}(\mathcal{Z})}} \mathrm{H}(\Phi \otimes \Psi(a \otimes b)) \\
&= \min_{a \in \mathrm{D}(\mathcal{X})} \mathrm{H}(\Phi(a)) + \min_{b \in \mathrm{D}(\mathcal{Z})} \mathrm{H}(\Psi(b)) \\
&=: \mathrm{H}_{\text{min}}(\Phi) + \mathrm{H}_{\text{min}}(\Psi)
\end{aligned}$$

since the minimum is over a strictly smaller set of density operators.

We must now show $\mathrm{H}_{\text{min}}(\Phi \otimes \Psi) \geq \mathrm{H}_{\text{min}}(\Phi) + \mathrm{H}_{\text{min}}(\Psi)$ to obtain the equality. To start we use the fact that $\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})}(\rho) \in \mathrm{SepD}(\mathcal{Y} : \mathcal{Z})$ for all complex Euclidean spaces $\mathcal{Z}$ and $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Z})$ as implied by $J(\Phi) \in \mathrm{Sep}(\mathcal{Y} : \mathcal{X})$ and problem 1. This allows us to write $\Phi \otimes \Psi(\rho)$ as a separable state as follows.

$$\begin{aligned}
\left[\Phi \otimes \Psi\right](\rho) &= \left(\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi\right) \underbrace{\left[\left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})}\right)(\rho)\right]}_{\in \mathrm{SepD}(\mathcal{Y} \otimes \mathcal{Z})} \\
&= \left(\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi\right) \sum_{a \in \Sigma} p(a)\, x_a x_a^* \otimes y_a y_a^* && \text{(Proposition 6.5)} \\
&= \sum_{a \in \Sigma} p(a) x_a x_a^* \otimes \Psi(y_a y_a^*)
\end{aligned}$$

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We can now calculate the entropy of such a state, and look to bound it below.

$$\begin{aligned}\mathrm{H}\big[(\Phi\otimes\Psi)(\rho)\big] &= \mathrm{H}\left[\sum_{a\in\Sigma} p(a)\, x_a x_a^* \otimes \Psi(y_a y_a^*)\right] \\ &\geq \sum_{a\in\Sigma} p(a)\,\mathrm{H}(x_a x_a^*) + \mathrm{H}\left[\sum_{a\in\Sigma} p(a)\Psi(y_a y_a^*)\right] \qquad \text{(HW 3 Problem 3 (b))}\end{aligned}$$

Taking the minimum of each side we have

$$\begin{aligned}&\min_{\rho\in\mathrm{D}(\mathcal{X}\otimes\mathcal{Z})} \mathrm{H}\big[(\Phi\otimes\Psi)(\rho)\big] \\ &\qquad\geq \min_{p\in\mathcal{P}(\Sigma)}\left(\min_{x_a\in\mathcal{S}(\mathcal{X})}\sum_{a\in\Sigma} p(a)\,\mathrm{H}(x_a x_a^*) + \min_{y_a\in\mathcal{S}(\mathcal{Y})}\mathrm{H}\left[\sum_{a\in\Sigma} p(a)\Psi(y_a y_a^*)\right]\right).\end{aligned}$$

Equation 7.398 from the text shows the minimum entropy is always achieved by a pure state. Applying this fact to the second term on the right hand side we must have one $\hat{a}\in\Sigma$ with $p(\hat{a}) = 1$, and $p(a) = 0$ otherwise. This simplifies the equality to

$$\min_{\rho\in\mathrm{D}(\mathcal{X}\otimes\mathcal{Z})} \mathrm{H}\big[(\Phi\otimes\Psi)(\rho)\big] \geq \min_{x\in\mathcal{S}(\mathcal{X})}\mathrm{H}(xx^*) + \min_{y\in\mathcal{S}(\mathcal{Y})}\mathrm{H}(\Psi(yy^*)).$$

With the $p(a)$'s out of the way, we then have $(\Phi\otimes\mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho) = x_{\hat{a}}x_{\hat{a}}^*\otimes y_{\hat{a}}y_{\hat{a}}^*$, and further

$$\Phi(\rho[\mathcal{X}]) = \mathrm{tr}_{\mathcal{Z}}\big[(\Phi\otimes\mathbb{1}_{\mathrm{L}(\mathcal{Z})})(\rho)\big] = x_{\hat{a}}x_{\hat{a}}^*.$$

We then have

$$\mathrm{H}_{\min}(\Phi\otimes\Psi) \geq \min_{x\in\mathcal{S}(\mathcal{X})}\mathrm{H}(\Phi(xx^*)) + \min_{y\in\mathcal{S}(\mathcal{Y})}\mathrm{H}(\Psi(yy^*)) = \mathrm{H}_{\min}(\Phi) + \mathrm{H}_{\min}(\Psi)$$

as desired. These two inequalities complete the proof.

Problem 3

Let $\mathcal{X}$ be a complex Euclidean space, let $n = \dim(\mathcal{X})$, and let $\Phi \in \mathrm{C}(\mathcal{X})$ be a unital channel. Following our usual convention for singular-value decompositions, let $s_1(Y) \geq \cdots \geq s_n(Y)$ denote the singular values of a given operator $Y \in \mathrm{L}(\mathcal{X})$, ordered from largest to smallest, and taking $s_k(Y) = 0$ when $k > \mathrm{rank}(Y)$. Prove that, for every operator $X \in \mathrm{L}(\mathcal{X})$, we have

$$s_1(X) + \cdots + s_m(X) \geq s_1(\Phi(X)) + \cdots + s_m(\Phi(X))$$

for every $m \in \{1, \ldots, n\}$.
Hint: thinking about the block operator

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = |0\rangle\langle 1| \otimes X + |1\rangle\langle 0| \otimes X^*$$

may be helpful when solving this problem.

Solution completed in collaboration with Mohammad Ayyash,[3] and Nicholas Zutt.[4]

Solution.

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Problem 4

Let Σ be an alphabet, let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces of the form $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Sigma$, define the swap operator

$$W = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|,$$

which we may regard as a unitary operator $W \in \mathrm{U}(\mathcal{X}\otimes\mathcal{Y})$, define projections

$$\Pi_0 = \frac{\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} + W}{2} \quad\text{and}\quad \Pi_1 = \frac{\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} - W}{2},$$

and define

$$\rho_0 = \frac{\Pi_0}{\binom{n+1}{2}} \quad\text{and}\quad \rho_1 = \frac{\Pi_1}{\binom{n}{2}},$$

for $n = |\Sigma|$. These are the symmetric and anti-symmetric Werner states that were discussed a few times, such as in Lecture 3. (See also Example 6.10 in the text.) Prove that if $\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{X}\otimes\mathcal{Y})$ is a measurement with $\mu(0), \mu(1) \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y})$ (i.e., μ is a *PPT measurement*), then

$$\frac{1}{2}\langle\mu(0)|\rho_0\rangle + \frac{1}{2}\langle\mu(1)|\rho_1\rangle \leq \frac{1}{2} + \frac{1}{n+1}.$$

Solution. Let's first simplify the terms on the left hand side as much as possible.

$$\begin{aligned}
\langle\mu(1)|\rho_1\rangle &= \frac{1}{\binom{n}{2}}\operatorname{tr}(\mu(1)^*\Pi_1) \\
&= \frac{1}{n(n-1)}\operatorname{tr}(\mu(1)(\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} - W)) \\
\langle\mu(0)|\rho_0\rangle &= \frac{1}{\binom{n+1}{2}}\operatorname{tr}(\mu(0)^*\Pi_0) \\
&= \frac{1}{n(n+1)}\operatorname{tr}(\mu(0)(\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} + W)) \\
&= \frac{1}{n(n+1)}\operatorname{tr}([\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} - \mu(1)]\,(\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} + W)) \\
&= \frac{1}{n(n+1)}\operatorname{tr}(\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} + W - \mu(1) - \mu(1)W) \\
&= \frac{1}{n(n+1)}\Big(n^2 + n - \operatorname{tr}(\mu(1) - \mu(1)W)\Big) \\
&= 1 - \frac{1}{n(n+1)}\operatorname{tr}(\mu(1)(\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} + W))
\end{aligned}$$

Putting these together, and doing some algebra we have the following equality.

$$\langle\mu(0)|\rho_0\rangle + \langle\mu(1)|\rho_1\rangle = 1 + \frac{2}{n(n^2-1)}\Big(\operatorname{tr}(\mu(1)) - n\operatorname{tr}(\mu(1)W)\Big)$$

Proving the desired inequality is now equivalent to showing $\operatorname{tr}(\mu(1)) - n\operatorname{tr}(\mu(1)W) \leq n(n-1)$. Substituting the second occurrence of $\mu(1)$ for $\mathbb{1}_\mathcal{X}\otimes\mathbb{1}_\mathcal{Y} - \mu(0)$ we obtain

$\operatorname{tr}(\mu(1)) - n^2 + n \operatorname{tr}(\mu(0)W)$ for the left hand side. Applying Hölder's inequality to $\langle \mu(0) | W \rangle$ with $p = 1$ and $p^* = \infty$ we see $\langle \mu(0) | W \rangle = \operatorname{tr}(\mu(0)W) \leq \|\mu(0)\|_1 \|W\|_\infty$. The swap operator has infinity norm equal to 1 (almost by definition), and $\|\mu(0)\|_1 = \operatorname{tr}\left(\sqrt{(\mu(0)^* \mu(0))}\right) = \operatorname{tr}(\mu(0))$. In total we then have $\operatorname{tr}(\mu(0)W) \leq \operatorname{tr}(\mu(0))$. Applying this to our desired inequality we have

$$\begin{aligned}
\operatorname{tr}(\mu(1)) - n^2 + n \operatorname{tr}(\mu(0)W) &= \operatorname{tr}(\mu(1)) - n^2 + (n-1)\operatorname{tr}(\mu(0)W) + \operatorname{tr}(\mu(0)W) \\
&\leq \operatorname{tr}(\mu(1)) - n^2 + (n-1)\operatorname{tr}(\mu(0)W) + \operatorname{tr}(\mu(0)) \\
&= \operatorname{tr}(\mu(0) + \mu(1)) - n^2 + (n-1)\operatorname{tr}(\mu(0)W) \\
&= (n-1)\operatorname{tr}(\mu(0)W).
\end{aligned}$$

Thus, in final we must show $\operatorname{tr}(\mu(0)W) \leq n$, or equivalently (by the completeness relation) $\operatorname{tr}(\mu(0)W) = \operatorname{tr}(W - \mu(1)W) = n - \operatorname{tr}(\mu(1)W)$ that $\operatorname{tr}(\mu(1)W) \geq 0$.

$$\begin{aligned}
\operatorname{tr}(\mu(1)W) &= \langle \mu(1) | W \rangle \\
&= \langle (\mathsf{T} \otimes \mathbb{1}_{\mathcal{Y}})\mu(1) | (\mathsf{T} \otimes \mathbb{1}_{\mathcal{Y}}) W \rangle \\
&= \left\langle A \middle| \sum_{a,b \in \Sigma} E_{a,b} \otimes E_{a,b} \right\rangle
\end{aligned}$$

Since $\mu(1) \in \mathrm{PPT}(\mathcal{X} : \mathcal{Y})$, we know $(\mathsf{T} \otimes \mathbb{1}_{\mathcal{Y}})\mu(1) \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$, and additionally it's not hard to show $\sum_{a,b \in \Sigma} E_{a,b} \otimes E_{a,b} \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$. Since we have the inner product of positive semidefinite operators, it must be a nonnegative real number. Thus $\operatorname{tr}(\mu(1)W) \geq 0$, and the proof is complete.