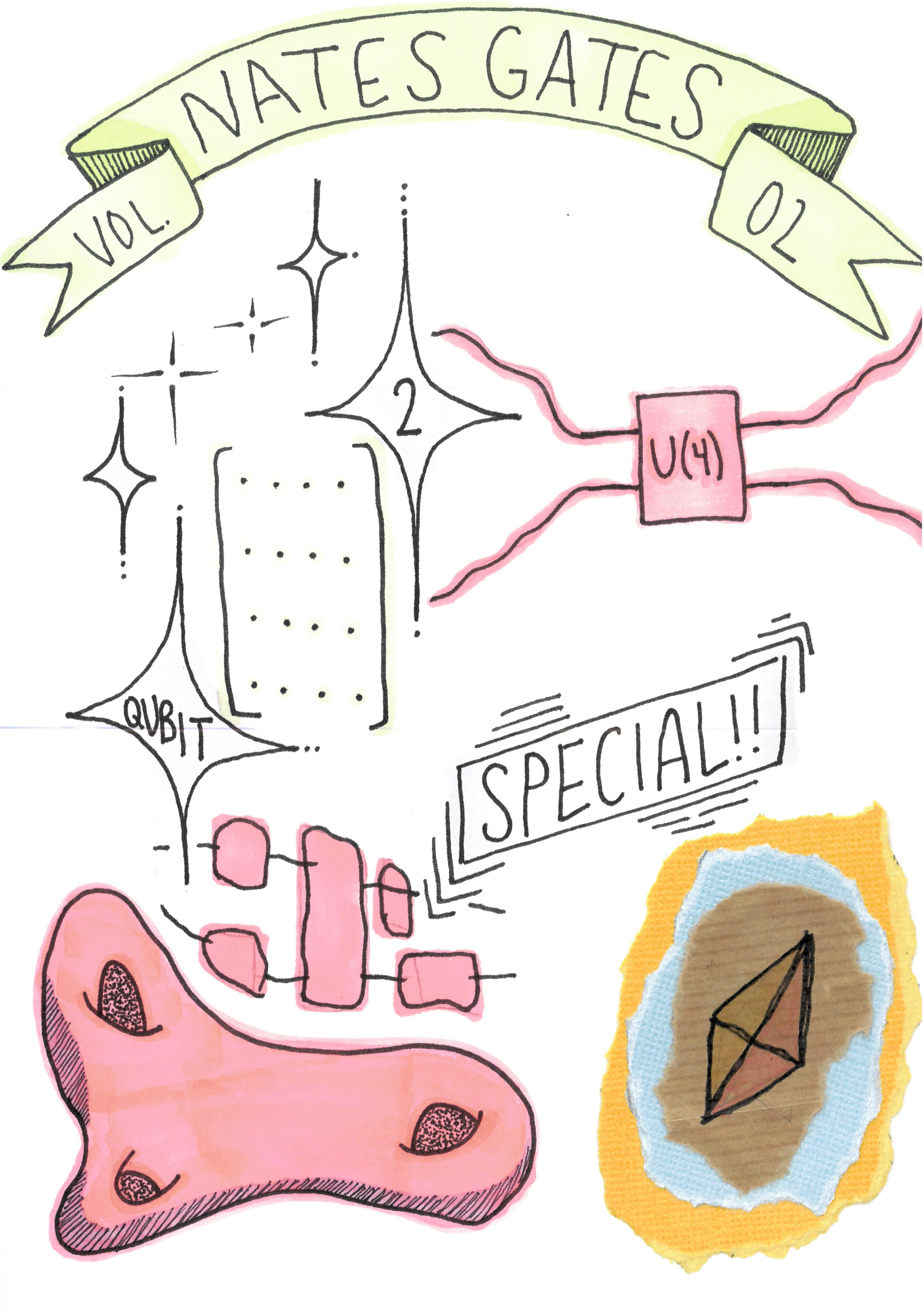
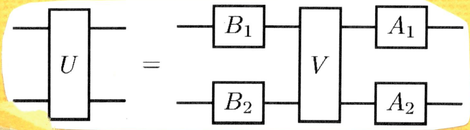
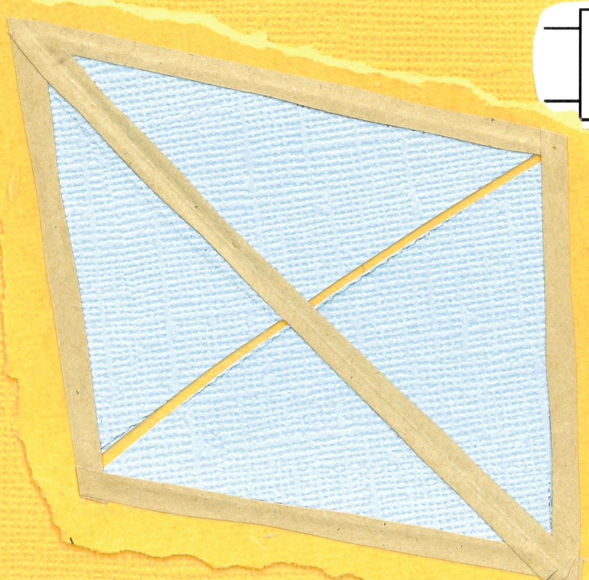

VOL.
NATES GATES
02
2
U(4)
QUBIT
SPECIAL!!

WEYL CHAMBER

$\mathsf{U}(n)$ is a complicated space. It has real dimension n^2 which means $\mathsf{U}(4)$ needs 16 real numbers to specify a single element. **16 number to specify a single operation on two qubits.** That's a lot! In an attempt to understand the structure of $\mathsf{U}(4)$ more concretely we might try to reduce the dimension to make it more concrete.

One potential way is to ask about the *entangling* structure of elements $U \in \mathsf{U}(4)$. Intuitively we might expect unitaries to have some local structure and some entangling structure. Since we only care about entanglement, lets drop the global phase and work in the 15-dimensional space $\mathsf{SU}(4)$. Next, define an equivalence relation on $\mathsf{U}(4)$ as $U \sim V$ if there exist $A_1, A_2, B_1, B_2 \in \mathsf{U}(2)$ such that





The quotient space $\mathsf{SU}(4)/\sim$ equates unitaries that equal up to single qubit operations. How much does this equivalence relation reduce the dimension of the space?

To answer this we'll use the following three facts:

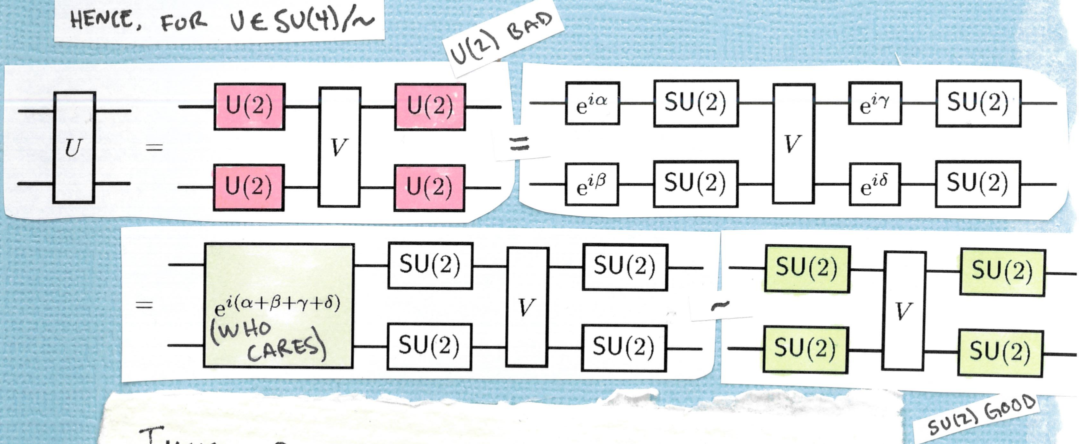
1. For all $U \in \mathsf{U}(n)$ there exists $\alpha \in \mathbb{R}$ and $\tilde{U} \in \mathsf{SU}(n)$ such that $U = \mathrm{e}^{i\alpha}\tilde{U}$.

   The underlying isomorphism: $\mathsf{U}(n) \cong (\mathsf{SU}(n) \times \mathsf{U}(1))/\mathbb{Z}_n$.

2. $A \otimes B = (\mathrm{e}^{i\alpha}A') \otimes (\mathrm{e}^{i\beta}B') = \mathrm{e}^{i(\alpha+\beta)}A' \otimes B'$ for $A, B \in \mathsf{U}(n)$ and $A', B' \in \mathsf{SU}(n)$.

3. Global phases commute with everything: $(\mathrm{e}^{i\alpha}\mathbb{1})U = U(\mathrm{e}^{i\alpha}\mathbb{1})$.

HENCE, FOR $U \in SU(4)/\sim$



THUS, BY DIMENSION COUNTING WE HAVE

$$\dim(SU(4)/\sim) = \dim(SU(4)) - 4\cdot\dim(SU(2))$$
$$= 15 - 4\cdot 3 = 3$$

We only need 3 real numbers to understand the entangling properties of *any* 2 qubit gate!

THE FOLLOWING GATE PARAMETRIZES $SU(4)/\sim$

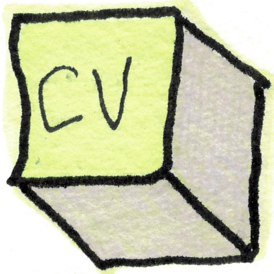
$$\mathrm{CAN}(a, b, c) := \exp(i\,(aX \otimes X + bY \otimes Y + cZ \otimes Z))$$

$$= \begin{bmatrix} \mathrm{e}^{ic}\cos(a-b) & 0 & 0 & i\,\mathrm{e}^{ic}\sin(a-b) \\ 0 & \mathrm{e}^{-ic}\cos(a+b) & i\,\mathrm{e}^{-ic}\sin(a+b) & 0 \\ 0 & i\,\mathrm{e}^{-ic}\sin(a+b) & \mathrm{e}^{-ic}\cos(a+b) & 0 \\ i\,\mathrm{e}^{ic}\sin(a-b) & 0 & 0 & \mathrm{e}^{ic}\cos(a-b) \end{bmatrix}$$

$\pi/4 \geq a \geq b \geq |c|$

THESE INEQUALITIES DEFINE THE POLYHEDRON ON THE PAGE PRIOR.

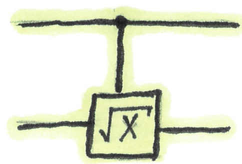
MORE DETAILS ON THREEPLUSONE.COM/PUBS/ON-GATES.PDF



CONTROLLED $\sqrt{X}$

[CV, CSX, C$\sqrt{X}$, ...]

$$CV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+i}{2} & \frac{1-i}{2} \\ 0 & 0 & \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} =$$



$\sim CAN(\frac{\pi}{8}, 0, 0)$

TO BE APPLIED WHEN $\sqrt{X}$ NEEDS SOME ADULT SUPERVISION.

SOME RECENT WORK [0] SUGGESTS QUANTUM ADVANTAGE MIGHT ARISE FROM THE ABILITY TO PERFORM CERTAIN $\sqrt{}$-OPERATIONS.

[0]: ARXIV:2602.16927

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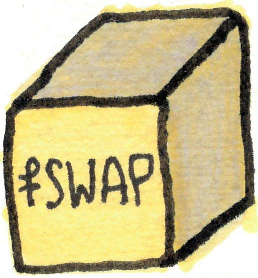
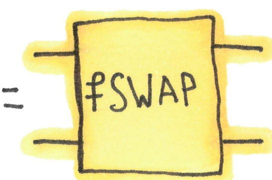
FERMIONIC SWAP



$$\mathrm{fSWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} =$$



$$\sim \mathrm{CAN}\left(\tfrac{\pi}{4}, \tfrac{\pi}{4}, 0\right)$$

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$\mathrm{SWAP}|\psi\rangle = a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle$$

$$\mathrm{fSWAP}|\psi\rangle = a|00\rangle + b|10\rangle + c|01\rangle - d|11\rangle$$

BASICALLY SWAP'S YOUNGER BROTHER THAT HAD TO DIFFERENTIATE THEMSELF.

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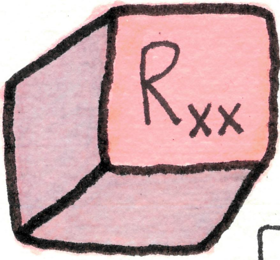
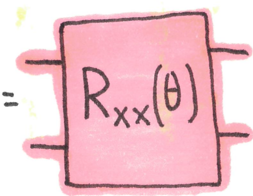


XX ROTATION

[R_{xx}, RXX, XX_θ, ...]

$$R_{xx}(\theta) = \begin{bmatrix} \cos\theta & 0 & 0 & -i\sin\theta \\ 0 & \cos\theta & -i\sin\theta & 0 \\ 0 & -i\sin\theta & \cos\theta & 0 \\ -i\sin\theta & 0 & 0 & \cos\theta \end{bmatrix} =$$



$$\sim CAN(\theta, 0, 0)$$

OKAY, MOST PEOPLE DEFINE R_{xx} WITH A $\frac{\theta}{2}$, BUT I GOT LIMITED REAL ESTATE. THE CANONICAL COORDINATES POP OUT OF THE FACT THAT

$$R_{xx}(\theta) = e^{-i\theta X \otimes X}$$

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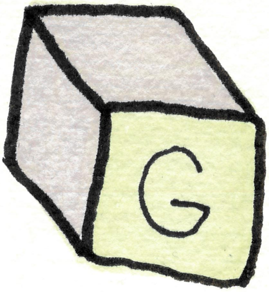
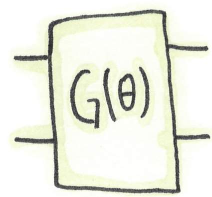


GIVENS ROTATION

$[G, G_{ij}, \ldots]$

$$G(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$



$$\sim \mathrm{CAN}\left(\frac{\theta}{2}, \frac{\theta}{2}, 0\right)$$

A rotation in the $|01\rangle, |10\rangle$ subspace.

Can be generalized to act only on basis states $|i\rangle, |j\rangle$. If i and j are ~~any~~ ALL basis states, then $\{G_{ij}(\theta)\}$ is a GIVIN' universal gate set!

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U(4)

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